# Practical Economic Statistical Design of $\overline{X}$ Chart

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**Abstract.** This paper developed an understandable economic mathematical model of total cost of quality related to the use of control chart. This model considers both statistical quality and cost criteria. The statistical quality constraints considered in this paper are type I error ($\alpha$) and type II error ($\beta$). The total cost of quality is a function of sample size ($n$) and sampling frequency ($f$). The model of total cost can be optimized to determine values of $n$ and $f$ that minimize the total cost of quality, while quality level is satisfied. The model can be solved by common spreadsheet program. The economic statistical model is developed under the real situation of a case study company to make the model realistic.

## Introduction

Statistical process control is an effective method for controlling quality. The primary tool of statistical process control is control chart. Control chart is used as a tool to monitor a quality characteristic of a process. It provides information of about the stability of a process in terms of process mean and variation. $\overline{X}$ chart is a commonly used tool in manufacturing industries for monitoring process mean.

The design of the $\overline{X}$ chart involves with the determination of sample size and sampling frequency. Traditionally, control charts have been designed with the consideration of statistical criteria only [1]. The problem of this approach is that cost effectiveness is not obtained. Many researchers have proposed economic models for designing of control chart. However, very few economic models for the design of control charts have been implemented [2]. The economic models are not widely used because the models are quite complex, and difficult to evaluate and optimize [3]. These models usually use complex mathematics and statistics, which are difficult to understand and apply in practice. In addition, control charts based on only economically optimal design generally have poor statistical properties [4]. Moreover, these models did not consider all relevant costs and no formal optimization techniques have been applied to the total cost function [1].

This research aims to solve the above-mentioned problems by developing an understandable economic mathematical cost model where statistical properties are taken into account. The model expresses the total cost of quality related to the use of control chart. This model considers both quality and cost criteria. The presented cost equations to estimate cost elements are simple. The total cost of quality is a function of sample size ($n$) and sampling frequency ($f$). The model of total cost can be optimized to find values of $n$ and $f$ that minimize the total cost of quality, while quality level is satisfied. The economic statistical model will be developed under the real situation of a case study company to make the model realistic.

The case study company is in electronic industry. It produces specific parts in a motor for hard disk drive. The production of the product consists of turning process, internal process quality assurance (IPQA), and outgoing quality assurance (OQA). Inside diameter is a quality characteristic to be monitored on an $\overline{X}$ chart by the IPQA staff. The OQA process is the final quality control process before delivering the product to customers. Variable sampling plans [5] are used to test the product lot both in IPQA and OQA processes. In the IPQA process, once an assignable cause is found, variable sampling plan is employed to decide on whether to accept or reject lot. If the lot is rejected, products need to be reworked and then re-inspected with another variable sampling plan. The customers also use variable sampling plan to test the incoming lot. They will claim for the new products in case that the product failed to the testing or failed to assembly in manufacturing process with penalty added.

## Cycle Time Determination

Generally in a production process, there is a cycle of in-control period, out-of-control period, and the period to take action to bring the process back to in-control state. The total cost in each cycle is assumed to be the same. Thus, the developed model expresses the total cost per hour unit, which is calculated from the total cost per cycle time divided by the cycle time.

Previous work proposed a production cycle, which consists of four time periods that are in-control period, out-of-control period, time to take a sample and interpret the result, and time to find an assignable cause [6]. However, this model does not allow the process to be shut down when searching for the assignable cause and it does not include the time and cost of repairing the process if it is found to be out-of-control [7].

This paper proposes that the cycle time should be divided into four stages as shown in Figure 1 and illustrated as follows. Process shut down and repairs are allowed in this model.

1. The in-control stage in $T_0$ time period. It is presumed that the process begins with a state of in-control. In this stage, the process is allowed to operate continuously with $\gamma_1$ probability of defective.

2. The delay detection stage in $ATS$ time period. The Average Time to Signal ($ATS$) is the mean time the control scheme takes to detect an out-of-control signal from the time of occurrence of the assignable cause or mean shift [8].

3. The finding an assignable cause stage in $T_1$ time period. Once the out-of-control signal is detected, the process is searched for an assignable cause in order to stop and repair if an assignable cause is found.

4. The repairing stage in $T_2$ time period. Once the assignable cause is found, the process is allowed to stop for repairing in $T_2$ time period.
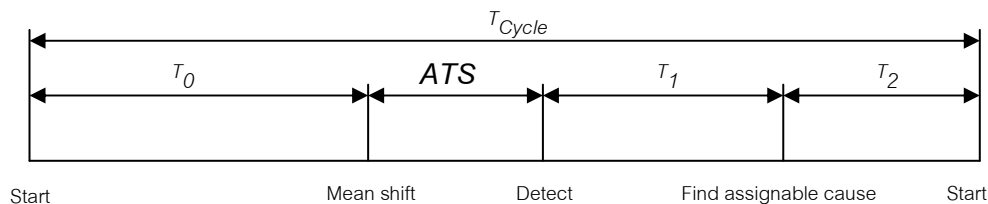


Fig. 1 The Cycle Time

$T_0$     = Time period that the process is in an in-control stage
$ATS$   = Average time between mean shift and its detection
$T_1$     = Time period for finding an assignable cause
$T_2$     = Time period that the process is stopped for repairing
$T_{Cycle}$ = The cycle time, where $T_{Cycle} = T_0 + ATS + T_1 + T_2$

## Cost Elements in the Proposed Model

In this paper, all cost elements are classified using the cost of quality concept to make the model more understandable [9]. The model expresses total cost of quality per hour unit associated with the use of control chart. Costs in the model consist of appraisal costs and failure costs. Prevention costs are excluded from the model because they are not depended on sample size and sampling frequency.

Appraisal costs are costs occurred from routine sampling activities. The appraisal costs consist of control chart sampling cost, false alarm cost, and OQA sampling cost. Control chart sampling cost is a cost occurred from sampling and testing activities to obtain data to be plotted on the control chart. False alarm cost is a cost of investigating process when there is an alarm in spite of the fact that the process is still in-control. OQA sampling cost occurs from sampling activities before delivering finished products to the customer.

Failure costs consist of internal and external failure costs. Internal failure costs occur from defective products found before delivering to customers and their consequences of creating loss in the company process. Internal failure costs consist of retest cost, defect cost, cease cost, and true

alarm cost. Retest cost occurs from retesting activities after out-of-control situation is signaled. Defect cost is a cost that occurs from scrapping defective products and activities needed for repairing products that can be reworked. Cease cost is a cost that occurs from an opportunity loss due to the stoppage of production process needed for repairing the process. True alarm cost is a cost that occurs from activities needed for repairing an assignable cause in the production process.

External failure costs are costs that occur after the products have been delivered to the customer. The external failure costs consist of transportation cost and replacement cost. Replacement cost is a cost of replacing the defective products with the new products and also the penalty cost from the customer. Transportation cost is a cost of transporting new products to replace the rejected defective products.

These cost elements are developed under the practice of the case study process. However, the cost elements in model can be adjusted to suit the process of each user.

This paper presents cost equations for the above-mentioned cost elements. The proposed cost model is developed with the intention to make them simple to calculate and understand. The parameter values need to be entered in the equations are commonly recorded or easily collected in any company. The notation of variables in the cost model is presented in the next section. Then, the optimization model and cost equations are subsequently presented.

**Notation**

The variables used in the cost model are explained as follows:

$n_i$ = Sample size (pieces/time)
$f_i$ = Sampling frequency (times/hour)
$P_{r\,i}$ = Production rate for each product (pieces/hour)
$M$ = Testing machine operating rate (baht/hour/machine)
$R_i$ = Rework rate (baht/piece)
$P_{ni}$ = Product profit (baht/piece)
$C_{Ti}$ = Cost of transportation to the customer (baht/time)
$O_{Ti}$ = Transportation lot size (pieces/time)
$I_{Ti}$ = Transportation interval (hour)
$C_{P\,IPQAi}$ = Product cost in IPQA process (baht/piece)
$C_{P\,OQAi}$ = Product cost in OQA process (baht/piece)
$C_{P\,Cusi}$ = Product cost when delivered to the customer (baht/piece)
$P_{eni}$ = Customer penalty rate (baht/piece)
$L_s$ = Staff labor rate (baht/hour/person)
$T_{Pick}$ = Time for collecting samples from each machine (hour-person/time)
$T_{Testi}$ = Time for testing a sample per piece (hour-person/piece)
$K_i$ = Testing machine capacity (piece/machine/hour), where $K_i = \dfrac{1}{T_{testi}}$
$D_i$ = Cost of destructed product (baht/piece)
$S_1$ = Engineer labor rate (baht/hour)
$S_2$ = Technician labor rate (baht/hour)
$n_{IPQAi}$ = Sample size in IPQA variable sampling plan (pieces/time)
$Y_i$ = Probability of rejecting the lot in IPQA variable sampling plan
$n_{IPQA\,ARi}$ = Sample size in IPQA AR variable sampling plan (pieces/time)
$H_i$ = Probability of rejecting the lot in IPQA AR variable sampling plan
$O_{Sizei}$ = OQA variable sampling plan lot size (pieces/time)
$n_{OQAi}$ = Sample size in OQA variable sampling plan (pieces/time)
$P_{0i}$ = Probability of rejecting the lot in OQA variable sampling plan
$n_{OQA\,Ari}$ = Sample size in OQA AR variable sampling plan (pieces/time)
$P_{di}$ = Probability of rejecting the lot in OQA AR variable sampling plan
$P_{reji}$ = Probability of rejecting the lot in customer variable sampling plan

$P_{mi}$ = Probability of defective found in the customer manufacturing process

$P_i$ = Percentage of defective product produced from IPQA process

$$P_i = \frac{[(ATS_i + T_{1i}) \times (\gamma_{2i} \times (1 - Y_i) + Y_i \times \gamma_{1i} \times (1 - H_i))] + [T_{0i} \times \gamma_{1i}]}{T_{Cyclei} - T_{2i}}$$

$P'_i$ = Percentage of defective product produced from OQA process

$$P'_i = P_i \times (1 - P_{0i}) + P_0 \times \gamma_{1i} \times (1 - P_{di})$$

$ATU$ = Number of testing machines used in the process (machine)

$$ATU = \sum_{i=1}^{m} (\frac{n_i \times f_i}{K_i}) + \sum_{i=1}^{m} (\frac{n_{OQAi} \times \Pr_i}{K_i \times O_{Sizei}}) + \sum_{i=1}^{m} (\frac{n_{IPQAi}}{K_i \times T_{Cyclei}}) + \sum_{i=1}^{m} (\frac{Y_i \times n_{IPQA\,ARi}}{K_i \times T_{Cyclei}}) + \sum_{i=1}^{m} \frac{(P_{oi} \times n_{OQAARi} \times \Pr_i)}{O_{Sizei} \times K_i}$$

$G_i$ = Number of testing machine available (machine)

### The Proposed Practical Economic Statistical Model

This section presents the model of total cost of quality associated with the use of $\bar{X}$ chart. The objective function and constraints are explained below.

In this paper, the control limit coefficient ($L$) is set to be 3 since three-sigma limits are customarily employed on control chart, regardless of the type of chart employed [5]. Therefore, with control limit coefficient of 3, the probability of false alarm ($\alpha$) or type I error is 0.0027 and it yields the in-control average run length ($ARL_0$) of 370 to retain the standard statistical property.

Another crucial statistical constraint for the design of control chart is the probability of not detecting the mean shift of size $k$ once the mean shift occurs. This is called type II error with probability $\beta$. The type II error can be expressed as out-of-control average run length ($ARL_1$) or average time to signal ($ATS$) where $ARL_1 = \frac{1}{1 - \beta}$ or $ATS = \frac{1}{1 - \beta} \times \frac{1}{f}$. The significant variables that affect $ATS$ are $\beta$ and $f$, while $\beta$ is affected by $n$ $\left(\beta = \Phi(L - k\sqrt{n}) - \Phi(-L - k\sqrt{n})\right)$. When sample size ($n$) and sampling frequency ($f$) are increased, $ATS$ is reduced. Shorter $ATS$ yields lower defective products and lower failure costs ($FC$). However, higher $n$ and $f$ cause higher appraisal costs ($AC$). Therefore, it is necessary to determine the levels of $n$ and $f$ that minimize the total cost ($TC$), which is the sum of $AC$ and $FC$. In practice, there are several products in the product set ($I$) that need to be sampled and share testing machines. Thus, it is required to determine the appropriate levels of $n$ and $f$ for each product, $i$, where $\forall i \in I$ that minimize the total cost without exceeding the capacity of the testing machines ($G$).

The problem of control chart design is formulated as follows:

Minimize $\quad TC = \sum_{i \in I} AC_i + FC_i$ (1)

such that

$\alpha_i = 0.0027 \quad \forall i \in I$ (2)

$\beta_i \leq 0.2 \quad \forall i \in I$ (3)

$\beta_i \geq 0 \quad \forall i \in I$ (4)

$n_i \geq 1 \quad \forall i \in I$ (5)

$f_i \leq \frac{1}{T_{picki}} \quad \forall i \in I$ (6)

$f_i \geq 0 \quad \forall i \in I$ (7)

$ATU_i \leq G_i \quad \forall i \in I$ (8)

The equation to calculate each cost element in the total cost of quality function is presented in Table 1.

Table 1. Equations of Cost Elements in the Proposed Model

| **Appraisal Costs (AC)** | |
|---|---|
| Control chart sampling cost | $L_S \times \sum_{i \in I}(T_{Pick} \times f_i + T_{Testi} \times n_i \times f_i) + M \times \sum_{i \in I}(\frac{n_i \times f_i}{K_i}) + \sum_{i \in I}(D_i \times n_i \times f_i)$ |
| False Alarm Cost | $\sum_{i \in I}(S_2 \times T_{1i} \times \alpha_i \times f_i \times \frac{T_{oi}}{T_{Cyclei}})$ |
| OQA Sampling Cost | $L_S \times \sum_{i \in I}\frac{(T_{Testi} \times n_{OQAi} \times Pr_i)}{O_{Sizei}} + M \times \sum_{i \in I}(\frac{n_{OQAi} \times Pr_i}{K_i \times O_{Sizei}}) + D_i \times \sum_{i \in I}\frac{(n_{OQAi} \times Pr_i)}{O_{Sizei}}$ |
| **Failure Costs (FC)** | |
| **Internal Failure Costs** | |
| True Alarm Cost | $\sum_{i \in I}\frac{(S_2 \times T_{1i} + S_1 \times T_{2i})}{T_{Cyclei}}$ |
| Cease Cost | $\sum_{i \in I}(\frac{T_{2i} \times Pr_i \times Pn_i}{T_{Cyclei}})$ |
| Retest Cost | |
| - IPQA Hold Lot Test Cost | $L_S \times \sum_{i \in I}(\frac{T_{Testi} \times n_{IPQAi}}{T_{Cyclei}}) + M \times \sum_{i \in I}(\frac{n_{IPQAi}}{K_i \times T_{Cyclei}}) + \sum_{i \in I}(\frac{D_i \times n_{IPQAi}}{T_{Cyclei}})$ |
| - IPQA after Rework Test Cost | $L_S \times \sum_{i \in I}(\frac{Y_i \times T_{Testi} \times n_{IPQA\,ARi}}{T_{Cyclei}}) + M \times \sum_{i \in I}(\frac{Y_i \times n_{IPQA\,ARi}}{K_i \times T_{Cyclei}}) + \sum_{i \in I}(\frac{Y_i \times D_i \times n_{IPQA\,ARi}}{T_{Cyclei}})$ |
| - OQA after Rework Test Cost | $L_S \times \sum_{i \in I}\frac{(P_{0i} \times T_{Testi} \times n_{OQAARi} \times Pr_i)}{O_{Sizei}} + M \times \sum_{i \in I}\frac{(P_{oi} \times n_{OQAARi} \times Pr_i)}{O_{Sizei} \times K_i} + \sum_{i \in I}\frac{(P_{oi} \times D_i \times n_{OQAARi} \times Pr_i)}{O_{Sizei}}$ |
| Defect Cost | |
| - IPQA Rework Cost | $\sum_{i \in I}\frac{Y_i \times R_i \times Pr_i \times (ATS_i + T_{1i})}{T_{Cyclei}}$ |
| - IPQA Scrap Cost | $\sum_{i \in I}\frac{Y_i \times H_i \times C_{P\,IPQAi} \times Pr_i \times (ATS_i + T_{1i})}{T_{Cyclei}}$ |
| - OQA Rework Cost | $R \times \sum_{i \in I}(P_{oi} \times Pr_i)$ |
| - OQA Scrap Cost | $\sum_{i \in I}(P_{oi} \times P_{di} \times C_{P\,OQAi} \times Pr_i)$ |

| External Failure Costs | |
|---|---|
| Transportation Cost | $\sum_{i \in I} C_{Ti} \times \left( \dfrac{roundup(\mathrm{Pr}_i \times (P_{reji} + (1 - P_{reji})(P_{mi})(P'_i)) \times \frac{I_{Ti}}{O_{Ti}})}{I_{Ti}} \right)$ |
| Replacement Cost | $\sum_{i \in I} (\mathrm{Pr}_i \times (P_{\mathrm{Re}\,ji} \times C_{Pcusi} + P_{mi} \times P'_i \times (1 - P_{reji}) \times (C_{Pcusi} + P_{eni})))$ |

## Optimization Technique

The main difficulties in the use of economic design are the computations involved [10]. The practical economic statistical model presented in this paper consists of mathematical formula that can be solved by Solver function in Microsoft Excel. This can help practitioners solve their problems without the difficulties of using complex program.

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