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Analytical Solution for Free Vibration of Multilayered Circular Plate

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Abstract. An analytical solution is deduced for a three-dimensional transversely isotropic axisymmetric multilayered circular plate made of piezoelectric (PE) and piezomagnetic (PM) under simply supported boundary condition. The state space vector, finite Hankel transform and propagation matrix methods are utilized together to obtain the full-field solutions for the multilayered circle plate. Numerical examples for five-layered PE/PM composites with dimensionless frequencies of the multilayered plate under simple-supported lateral boundary conditions are presented. The frequencies increase with the ratio of the thickness to radius.

1. Introduction

Magnetoelectroelastic coupling effect exists in multiphase materials which possess the ability of transforming energy from one form to another (among elastic, electric and magnetic forms). Intensive studies on the physical and mechanical properties of the structures were carried out by means of analytical, numerical and experimental methods. Among those structures, the composites made of piezoelectric (PE) and piezomagnetic (PM) layers were most frequently considered and some full-field exact solutions of these structures under certain boundary conditions were obtained. Pan and colleagues [1, 2] derived the static and free vibration solutions for multilayered rectangular plates under simply-supported boundary conditions by using the Stroh formulism and propagation matrix methods. By applying the state vector approach and propagation matrix method, Wang et al [3] derived the exact solution of the multilayered plate under static deformation, Chen et al [4] extended the static solution to the vibration case, and Chen et al [5] discussed the modal analysis of the multilayered plates. Combining the discrete layer approach and Ritz method, Ramirez et al [6] derived an approximate solution for the free vibration problem of two-dimensional laminate under both simply supported and fixed boundary conditions.

In this paper, starting from the equilibrium equations for each homogeneous layer and making use of the geometric and constitutive equations. By use of finite Hankel transform and let the free terms derived from the transform be zero. Base on the solutions of the state equations, the frequency equations are derived using the propagation matrix method and the boundary conditions on the bottom and top surfaces of the plates.

2. Basic equations

For a transversely isotropic coupling solid with its material axis parallel to z-axis, the general basic equation can be expressed as:

$$\sigma_i = c_{ik}\gamma_k - e_{ki}E_k - q_{ki}H_k \,,\; D_i = e_{ik}\gamma_k + \varepsilon_{ik}E_k + d_{ik}H_k \,,\; B_i = q_{ik}\gamma_k + d_{ik}E_k + \mu_{ik}H_k \tag{1}$$

$$\gamma_{ij} = 0.5\left(u_{i,j} + u_{j,i}\right),\quad E_i = -\varphi_{,i},\quad H_i = -\psi_{,i} \tag{2}$$

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2},\quad D_{j,j} = 0,\quad B_{j,j} = 0 \tag{3}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{D}$ and $\boldsymbol{B}$ are vectors of elastic stress, electric displacement and magnetic induction; γ, E and H are elastic strain, electric field and magnetic field; C, e, q, ε, μ and α, are respectively matrices of elastic stiffness, PE coefficients, PM coefficients, permittivity coefficients, permeability coefficients and magnetoelectric (ME) coefficients, where u, w, ϕ and ψ are, respectively, the displacement in r- and z-directions, and the electric and magnetic potentials; ρ is the density of the material.

The multilayered circular plate model is shown in Fig. 1

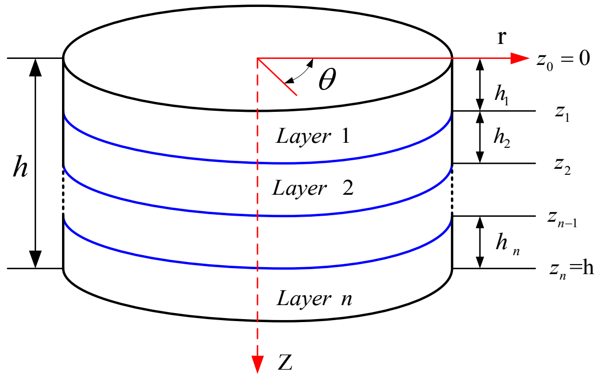

Figure 1. An-layered circular plate with each layer being either PE or PM material

3. General Solutions

Following[7], the dimensionless variables are defined as follows

$$\begin{cases} \xi = r/a,\quad \zeta = z/h,\quad \bar{u} = u/h,\quad \bar{w} = w/h,\quad \bar{\varphi} = \varphi\sqrt{\varepsilon_{33}^{(1)}/c_{11}^{(1)}}/h,\quad \bar{\psi} = \psi\sqrt{\mu_{33}^{(1)}/c_{11}^{(1)}} \\ \bar{\sigma}_{\mathrm{r}} = \sigma_{\mathrm{r}}/c_{11}^{(1)},\quad \bar{\sigma}_{\mathrm{z}} = \sigma_{\mathrm{z}}/c_{11}^{(1)},\quad \bar{\sigma}_{\theta} = \sigma_{\theta}/c_{11}^{(1)},\quad \bar{\sigma}_{\mathrm{rz}} = \sigma_{\mathrm{rz}}/c_{11}^{(1)},\quad \bar{D}_{\mathrm{r}} = D_{\mathrm{r}}/\sqrt{\varepsilon_{33}^{(1)}c_{11}^{(1)}} \\ \bar{D}_{\mathrm{z}} = D_{\mathrm{z}}/\sqrt{\varepsilon_{33}^{(1)}c_{11}^{(1)}},\quad \bar{B}_{\mathrm{r}} = B_{\mathrm{r}}/\sqrt{\mu_{33}^{(1)}c_{11}^{(1)}},\quad \bar{B}_{\mathrm{z}} = B_{\mathrm{z}}/\sqrt{\mu_{33}^{(1)}c_{11}^{(1)}},\quad \Omega = \omega h\sqrt{\rho/c_{11}^{(1)}} \end{cases} \tag{4}$$

where h and a are, respectively, total thickness and radius of the plate; $c_{11}^{(1)}$, $\varepsilon_{11}^{(1)}$ and $\mu_{11}^{(1)}$ are, respectively, the elastic stiffness, electric permittivity and magnetic permeability constants of the first layer material; Ω is the dimensionless frequency of the multilayered circular plate.
By choosing the primary variables as the state space vector and rearranging Eqs. (1), (2) and (3), we arrive at the following state equation

$$\frac{\partial \bar{\boldsymbol{R}}(\xi,\zeta)}{\partial \zeta} = \boldsymbol{A}\bar{\boldsymbol{R}}(\xi,\zeta) \tag{5}$$

$$\bar{\boldsymbol{R}}(\xi,\zeta) \equiv \left[\bar{u}(\xi,\zeta)\quad \bar{\sigma}_{\mathrm{z}}(\xi,\zeta)\quad \bar{D}_{\mathrm{z}}(\xi,\zeta)\quad \bar{B}_{\mathrm{z}}(\xi,\zeta)\quad \bar{\sigma}_{\mathrm{rz}}(\xi,\zeta)\quad \bar{w}(\xi,\zeta)\quad \bar{\varphi}(\xi,\zeta)\quad \bar{\psi}(\xi,\zeta)\right]^{\mathrm{T}} \tag{6}$$

where $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{F} \\ \mathbf{G} & \mathbf{0} \end{bmatrix}$ is the 8×8 operator matrix.

$$\mathbf{F}=\begin{bmatrix} f_1 & -s\frac{\partial}{\partial\xi} & f_2\frac{\partial}{\partial\xi} & f_3\frac{\partial}{\partial\xi} \\ -s\left(\frac{1}{\xi}+\frac{\partial}{\partial\xi}\right) & -\Omega^2 & 0 & 0 \\ f_2\left(\frac{1}{\xi}+\frac{\partial}{\partial\xi}\right) & 0 & f_4\left(\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\xi^2}\right) & f_5\left(\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\xi^2}\right) \\ f_3\left(\frac{1}{\xi}+\frac{\partial}{\partial\xi}\right) & 0 & f_5\left(\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\xi^2}\right) & f_6\left(\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\xi^2}\right) \end{bmatrix}, \mathbf{G}=\begin{bmatrix} -\Omega^2+g_1\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}-\frac{1}{\xi^2}\right) & g_2\frac{\partial}{\partial\xi} & g_3\frac{\partial}{\partial\xi} & g_4\frac{\partial}{\partial\xi} \\ g_2\left(\frac{1}{\xi}+\frac{\partial}{\partial\xi}\right) & g_5 & g_6 & g_7 \\ g_3\left(\frac{1}{\xi}+\frac{\partial}{\partial\xi}\right) & g_8 & g_9 & g_{10} \\ g_4\left(\frac{1}{\xi}+\frac{\partial}{\partial\xi}\right) & g_7 & g_{10} & g_{11} \end{bmatrix} \quad (7)$$

The related parameters are list in Appendix.

Define the following finite Hankel transform

$$J_\mu(f(\xi,\zeta))=\int_0^1 \xi f(\xi,\zeta)J_\mu(k\xi)d\xi \quad (8)$$

where $J_\mu(k\xi)$ is the first kind of Bessel function of μth order.

Applying the Hankel transform to both sides of Eq.(5), we obtain

$$\frac{\partial \boldsymbol{R}(k,\zeta)}{\partial\zeta}=\boldsymbol{K}(k)\boldsymbol{R}(k,\zeta)+\boldsymbol{Q}(k,\zeta) \quad (9)$$

where

$$\boldsymbol{K}(k)=\begin{bmatrix} 0 & 0 & 0 & 0 & f_1 & sk & -f_2k & -f_3k \\ 0 & 0 & 0 & 0 & -sk & -\Omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_2k & 0 & -f_4k^2 & -f_5k^2 \\ 0 & 0 & 0 & 0 & f_3k & 0 & -f_5k^2 & -f_6k^2 \\ -\Omega^2-g_1k^2 & -g_2k & -g_3k & -g_4k & 0 & 0 & 0 & 0 \\ g_2k & g_5 & g_6 & g_7 & 0 & 0 & 0 & 0 \\ g_3k & g_8 & g_9 & g_{10} & 0 & 0 & 0 & 0 \\ g_4k & g_7 & g_{10} & g_{11} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (10)$$

And eliminate Q according to boundary condition.

4. Load boundary conditions

We now consider the simply supported lateral boundary condition at ξ=1, i.e.,

$$\overline{w}(1,\zeta)=0, \overline{\phi}(1,\zeta)=0, \overline{\psi}(1,\zeta)=0, J_0(k)=0 \text{ and } \frac{c_{11}-c_{12}}{c_{11}^{(1)}}s\overline{u}(1,\zeta)+\overline{\sigma}_r(1,\zeta)=0 \quad (11)$$

Then Eq. (9) becomes a homogeneous set of equations. The solution can be assumed as the following exponential form

$$\boldsymbol{R}(k,\zeta)=\exp(\boldsymbol{K}(k)\zeta)\boldsymbol{R}(k,0) \quad (12)$$

is called the propagator matrix and varies along the vertical coordinate. It relates the values of space vector at arbitrary height ζ to that at the top surface of the layer.

Considering the continuity on the layer interface of two adjacent layers, say at $z=z_j$ between layers j and j+1, then space vectors satisfies the following relation

$$\boldsymbol{R}_{j+1}(k,0)=\boldsymbol{R}_j(k,h_j/h), \quad j=1,2,...,N \quad (13)$$

Making use of the boundary conditions on the top and bottom surfaces of the layered plate, the space vectors can be related by

$$\boldsymbol{R}_n(k,h_n/h)=\boldsymbol{F}(k)\boldsymbol{R}_1(k,0) \quad (14)$$

$$\boldsymbol{F}(k)=\prod_{j=1}^{N}\boldsymbol{T}_j(k,h_j/h) \tag{15}$$

The corresponding frequency equation can be obtained from Eq. (15)

$$\det(\mathbf{F})=0 \tag{16}$$

The frequency equation is transcendental with respect to Ω_i and gives infinite number of frequencies for each *k*.

5. Numerical examples

Firstly, we consider the multilayered circular plate structures composed of PE material $BaTiO_3$ and PM material $CoFe_2O_4$. Following the convention (i.e., B= $BaTiO_3$, and F= $CoFe_2O_4$), we use BFBFB for the $BaTiO_3/CoFe_2O_4/BaTiO_3/CoFe_2O_4/BaTiO_3$ plate and FBFBF for the $CoFe_2O_4/BaTiO_3/CoFe_2O_4/BaTiO_3/CoFe_2O_4$ plate. The total thickness is assumed to be 20mm with each layer at 4mm. The BFBFB and FBFBF five dimensionless frequencies and the corresponding *k1* , *k2* and *k3* for the simply supported lateral boundary condition with different thickness-to-radius ratio *s* are shown in Table 2 and 3. It shows that the dimensionless frequency increase with the thickness-to-radius ratio *s* for the simply supported lateral boundary condition.

6. Conclusions

We derived an analytical solution for three-dimensional transversely isotropic axisymmetric multilayered circular plates made of piezoelectric (PE) and piezomagnetic (PM) under simply supported lateral boundary condition. The finite Hankel transform and propagation matrix methods are utilized to find the full-field solutions for the multilayered circle plate. Numerical examples for five-layered PE/PM composites with dimensionless frequencies of the multilayered plate under simple-supported lateral boundary conditions are presented and discussed. The frequencies increase with the raitos of thickness to radius. These results can be served as benchmark solutions for future numerical analyses of multilayered circle plates.

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Appendix

$$s = h/a,\quad f_1 = c_{11}^{(1)}/c_{44},\quad f_2 = -\frac{se_{15}}{c_{44}\sqrt{\varepsilon_{33}^{(1)}/c_{11}^{(1)}}},\quad f_3 = -\frac{sq_{15}}{c_{44}\sqrt{\mu_{33}^{(1)}/c_{11}^{(1)}}},$$

$$f_4 = s^2(e_{15}^2/c_{44}+\varepsilon_{11})/\varepsilon_{33}^{(1)},\quad f_5 = s^2(e_{15}q_{15}/c_{44}+\alpha_{11})/\sqrt{\mu_{33}^{(1)}\varepsilon_{33}^{(1)}},$$

$$f_6 = s^2(q_{15}q_{15}/c_{44}+\mu_{11})/\mu_{33}^{(1)},\quad g_0 = c_{33}\varepsilon_{33}\mu_{33}+e_{33}e_{33}\mu_{33}+q_{33}\varepsilon_{33}q_{33},$$

$$g_1 = \frac{s^2}{c_{11}^{(a)}g_0}\Big(e_{33}^2(q_{13}^2+c_{11}\mu_{33})-e_{13}e_{33}(q_{13}q_{33}+c_{13}\mu_{33})+e_{13}^2(q_{33}^2+c_{33}\mu_{33}) + \varepsilon_{33}(c_{33}q_{13}^2-2c_{13}q_{13}q_{33}+c_{11}q_{33}^2-c_{13}^2\mu_{33}+c_{11}c_{33}\mu_{33})\Big),$$

$$g_2 = s(-c_{13}\varepsilon_{33}\mu_{33}-e_{31}e_{33}\mu_{33}-q_{31}q_{33}\varepsilon_{33})/g_0,$$

$$g_3 = s(c_{33}e_{31}\mu_{33}-c_{13}e_{33}\mu_{33}-e_{33}q_{31}q_{33}+e_{31}q_{33}^2)\sqrt{\varepsilon_{33}^{(1)}/c_{11}^{(1)}}/g_0,$$

$$g_4 = -s(-e_{33}^2q_{31}-c_{33}\varepsilon_{33}q_{31}+e_{31}e_{33}q_{33}+c_{13}\varepsilon_{33}q_{33})\sqrt{\mu_{33}^{(1)}/c_{11}^{(1)}}/g_0,$$

$$g_5 = c_{11}^{(1)}\varepsilon_{33}\mu_{33}/g_0,\quad g_6 = e_{33}\mu_{33}\sqrt{c_{11}^{(1)}\varepsilon_{33}^{(1)}}/g_0,\quad g_7 = \varepsilon_{33}q_{33}\sqrt{c_{11}^{(1)}\mu_{33}^{(1)}}/g_0,\quad g_8 = e_{33}\mu_{33}\sqrt{c_{11}^{(1)}\varepsilon_{33}^{(1)}}/g_0,$$

$$g_9 = -(c_{33}\mu_{33}+q_{33}^2)\varepsilon_{33}^{(1)}/g_0,\quad g_{10} = e_{33}q_{33}\sqrt{\mu_{33}^{(1)}\varepsilon_{33}^{(1)}}/g_0,\quad g_{11} = -(e_{33}^2+c_{33}\varepsilon_{33})\mu_{33}^{(1)}/g_0,$$

$$l_1 = \frac{s^2}{c_{11}^{(a)}g_0}\Big(e_{33}^2(q_{13}^2+c_{12}\mu_{33})-e_{13}e_{33}(q_{13}q_{33}+c_{13}\mu_{33})+e_{13}^2(q_{33}^2+c_{33}\mu_{33}) + \varepsilon_{33}(c_{33}q_{13}^2-2c_{13}q_{13}q_{33}+c_{12}q_{33}^2-c_{13}^2\mu_{33}+c_{12}c_{33}\mu_{33})\Big)$$

Table 1. Material properties of $BaTiO_3$ and $CoFe_2O_4$ [1] (C_{ij}: elastic constants in GPa; e_{ij}: piezoelectric coefficients in N/(V·m); q_{ij}: piezomagnetic coefficients in N/(A·m); ε_{ij}: permittivity coefficients in 10^{-9} C/(V·m); and μ_{ij}: permeability coefficients in 10^{-6} Wb/(A·m).

$BaTiO_3$	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}
	166	77	78	166	78	162	43	43	44.5
	e_{13}	e_{23}	e_{33}	e_{42}	e_{51}	ε_{11}	ε_{22}	ε_{33}	
	-4.4	-4.4	18.6	11.6	11.6	11.2	11.2	12.6	
$CoFe_2O_4$	C_{11}	C_{12}	C_{13}	C_{22}	C_{23}	C_{33}	C_{44}	C_{55}	C_{66}
	286	173	170.5	286	170.5	269.5	45.3	4 5.3	56.5
	q_{13}	q_{23}	q_{33}	q_{42}	q_{51}	μ_{11}	μ_{22}	μ_{33}	
	580.3	580.3	699.7	550	550	590	590	157	

Table 2. BFBFB dimensionless frequencies of the magnetoelectroelastic plate ($\Omega = \omega h\sqrt{\rho / c_{11}^{(1)}}$)

s	k1=2.40483			k2=5.52008			k3=8.65373		
	Ω1	Ω2	Ω3	Ω1	Ω2	Ω3	Ω1	Ω2	Ω3
0.1	0.01562	0.23272	1.71047	0.07888	0.53281	1.79070	0.18151	0.83130	1.91611
0.2	0.06061	0.46460	1.76793	0.27844	1.05485	2.03336	0.57808	1.61408	2.39830
0.3	0.13031	0.69477	1.85390	0.54010	1.55037	2.35170	1.03827	2.24727	2.96779
0.4	0.21912	0.92223	1.96164	0.82860	1.99147	2.70765	1.51217	2.63614	3.54627
0.5	0.32189	1.14572	2.08594	1.12765	2.33879	3.07842	1.98629	2.90999	4.08241
0.6	0.43452	1.36361	2.22270	1.43014	2.58258	3.44812	2.45745	3.18860	4.54287
0.7	0.55395	1.57381	2.36869	1.73299	2.76742	3.80358	2.92541	3.49932	4.93070
0.8	0.67803	1.77356	2.52136	2.03501	2.93759	4.13367	3.39072	3.84164	5.27807
0.9	0.80524	1.95947	2.67870	2.33572	3.11363	4.43144	3.85403	4.20973	5.61486
1.0	0.93449	2.12786	2.83905	2.63509	3.30232	4.69688	4.31590	4.59787	5.95722

Table 3. FBFBF dimensionless frequencies of the magnetoelectroelastic plate ($\Omega = \omega h\sqrt{\rho / c_{11}^{(1)}}$)

s	k1=2.40483			k2=5.52008			k3=8.65373		
	Ω1	Ω2	Ω3	Ω1	Ω2	Ω3	Ω1	Ω2	Ω3
0.1	0.01673	0.23826	1.74117	0.08421	0.54523	1.82013	0.19293	0.84984	1.94877
0.2	0.06476	0.47551	1.79730	0.29488	1.07714	2.07089	0.60698	1.63950	2.45178
0.3	0.13880	0.71065	1.88451	0.56761	1.57610	2.40330	1.08091	2.25530	3.03636
0.4	0.23256	0.94240	1.99607	0.86555	2.00972	2.77142	1.56539	2.63715	3.60805
0.5	0.34039	1.16923	2.12583	1.17249	2.34294	3.14772	2.04818	2.92896	4.11292
0.6	0.45793	1.38910	2.26877	1.48170	2.58234	3.51301	2.52665	3.22994	4.53701
0.7	0.58198	1.59946	2.42098	1.79045	2.77525	3.85316	3.00067	3.56007	4.90375
0.8	0.71036	1.79716	2.57944	2.09771	2.95885	4.16019	3.47054	3.91168	5.24441
0.9	0.84151	1.97873	2.74170	2.40315	3.14932	4.43400	3.93660	4.29381	5.57986
1.0	0.97442	2.14104	2.90575	2.70674	3.35153	4.68087	4.39903	4.68552	5.92080