Embedding Complete Binary Trees into Locally Twisted Cubes

Yuejuan Han1,2,a, Jianxi Fan2,b, Lantao You3,2,c, Yan Wang1,2,d
1Provincial Key Laboratory for Computer Information Processing Technology, Soochow University
2School of Computer Science and Technology, Soochow University
3Suzhou Industrial Park Institute Of Services Outsourcing, Suzhou 215006, China
a hyj@suda.edu.cn, b jxfan@suda.edu.cn, c yoult@siso.edu.cn, d wangyanme@suda.edu.cn

Keywords: Interconnection networks; locally twisted cube; complete binary tree; embedding; parallel computing.

Abstract. The locally twisted cube is a newly introduced interconnection network for parallel computing, which possesses many desirable properties. In this paper, the problem of embedding complete binary trees into locally twisted cubes is studied. Let $LTQ_n(V,E)$ denote the $n$-dimensional locally twisted cube. We find the following result in this paper: for any integer $n \geq 2$, we show that a complete binary tree with $2^n - 1$ nodes can be embedded into the $LTQ_n$ with dilation 2.

1. Introduction

An interconnection network can be represented by a graph $G = (V,E)$, where $V$ represents the node set and $E$ represents the edge set. One of the important properties of interconnection networks is graph embedding ability. Given a host graph $G_2 = (V_2,E_2)$, which represents the network into which other networks are to be embedded, and a guest graph $G_1 = (V_1,E_1)$, which represents the network to be embedded, the problem is to find a mapping from each node of $G_1$ to a node of $G_2$, and a mapping from each edge of $G_1$ to a path in $G_2$. Two common measures of effectiveness of an embedding are the dilation and expansion. The dilation of embedding $\psi$ is defined as $dil(G_1,G_2,\psi) = \max\{\text{dist}(G_2,\psi(u),\psi(v))|(u,v) \in E_1\}$, where $\text{dist}(G_2,\psi(u),\psi(v))$ denotes the distance between the two nodes $\psi(u)$ and $\psi(v)$ in $G_2$. The smaller the dilation of an embedding is, the shorter the communication delay that the graph $G_2$ simulates the graph $G_1$ [1]. The expansion of embedding is defined as $\exp(G_1,G_2,\psi) = |V(G_2)|/|V(G_1)|$, which measures the processor utilization. The smaller the expansion of an embedding is, the more efficient the processor utilization that the graph $G_2$ simulates the graph $G_1$. Graph embedding has good applications in transplanting parallel algorithms developed for one network to a different one, and allocating concurrent processes to processors in the network. Path, cycle and mesh are three fundamental networks for parallel computing, and much work about path, cycle and mesh embedding [3],[4],[6] appeared in the literature.

Trees are another common interconnection structures used in parallel computing. It is important to study the problem of how to embed different kinds of trees into a host graph. Recently, many tree embedding problems have been studied [9],[7],[10].

The locally twisted cube $LTQ_n$ is a variant of hypercube, proposed by Yang et al. [11]. It has many attractive features superior to those of the hypercube, such as the diameter is only about half of that of $Q_n$. In particular, Yang et al. [12] showed that $LTQ_n$ is Hamiltonian connected and contains a cycle of every length from 4 to $2^n$ for $n \geq 3$. Furthermore, $LTQ_n$ was proved to be $(n-2)$-pancyclic [2], for any integer $n \geq 3$. And some other properties of locally twisted cubes were discussed [8],[7],[5].

In this paper, the problem of embedding complete binary trees into locally twisted cubes is studied. We find for any integer $n \geq 2$, a complete binary tree with $2^n - 1$ nodes can be embedded into the $LTQ_n$ with dilation 2.
2. Preliminaries

A binary string \( x \) of length \( n \) is denoted by \( x_1x_2\ldots x_{n-1}x_n \), where \( x_1 \) is the most significant bit and \( x_n \) is the least significant bit.

Similar to \( Q_n \), \( LTQ_n \) is an \( n \)-regular graph of \( 2^n \) nodes. Every node of \( LTQ_n \) is identified by a unique binary string of length \( n \). \( LTQ_n \) can be recursively defined as follows.

\[
\begin{align*}
\text{(1) } & LTQ_2 \text{ is a graph consisting of four nodes labeled with 00, 01, 10, and 11, respectively, connected by four edges (00, 01), (00, 10), (01, 11), and (10, 11).} \\
\text{(2) For } n \geq 3, & LTQ_n \text{ is built from two disjoint copies of } LTQ_{n-1} \text{ with the following steps. Let } LTQ_{n-1}^0 \text{ denote the graph obtained by prefixing the label of each node of one copy of } LTQ_{n-1} \text{ with 0, and } LTQ_{n-1}^1 \text{ denote the graph obtained by prefixing the label of each node of the other copy of } LTQ_{n-1} \text{ with 1. Connect each node } x = 0x_2x_3\ldots x_n \text{ of } LTQ_{n-1}^0 \text{ to the node } 1(x_2 + x_n)x_3\ldots x_n \text{ of } LTQ_{n-1}^1 \text{ with an edge, where ' + ' represents the modulo 2 addition.}
\end{align*}
\]

We use \( CBT_n \) to denote the complete binary tree with \( 2^n - 1 \) nodes which can be embedded into \( LTQ_n \). For any integer \( i \in \{0, 1\} \), \( CBT^i_n \) denotes to prefix the node labels of \( CBT_n \) with \( i \). \( |P| \) is the length of path \( P \).

3. Embedding complete binary trees into locally twisted cubes

**Lemma 1.** A complete binary tree with 3 nodes can be embedded into \( LTQ_2 \) with dilation 1 rooted at any node of \( LTQ_2 \).

**Proof.** Obviously, we can embedded a complete binary tree rooted at any node of \( LTQ_2 \) into \( LTQ_2 \) with dilation 1, the lemma holds. \( \square \)

Considering the symmetric properties of \( LTQ_3 \), we can intuitively find the symmetric properties of \( LTQ_3 \) as shown in the following lemma.

**Lemma 2.** Let \( f_1, f_2, f_3 \) be three self-isomorphic mappings from \( V(LTQ_3) \) to \( V(LTQ_3) \) as follows:

\[
\begin{align*}
\text{(1) } & f_1(0x_2x_3) = f_1(1x_2\bar{x}_3) \text{ and } f_1(1x_2x_3) = f_1(0\bar{x}_2\bar{x}_3); \\
\text{(2) } & f_2(0x_2x_3) = f_2(0x_2\bar{x}_3) \text{ and } f_2(1x_2x_3) = f_2(1\bar{x}_2\bar{x}_3); \\
\text{(3) } & f_3(x_1x_2x_3) = f_3(x_1\bar{x}_2x_3); \quad \text{where } x_1, x_2, x_3 \in \{0, 1\}.
\end{align*}
\]

**Lemma 3.** A complete binary tree with 7 nodes can be embedded into \( LTQ_3 \) with dilation 1 rooted at any node of \( LTQ_3 \).

**Proof.** By Lemma 2, we only need to consider the following two nodes: 000 and 100. Figure 1 demonstrates two complete binary trees which can be embedded into \( LTQ_3 \) rooted at 000 and 100, respectively. It is easy to verify that a complete binary tree can be embedded into \( LTQ_3 \) rooted at any node of \( LTQ_3 \) with dilation 1, see Figure 1, every node of \( CBT_3 \) is mapping to a node in \( LTQ_3 \) and every edge of \( CBT_3 \) is mapping to an edge in \( LTQ_3 \), the lemma holds. \( \square \)
Lemma 4. A complete binary tree with 15 nodes can be embedded into $LTQ_4$ with dilation 1 rooted at $1000$, while another complete binary tree with 15 nodes can be embedded into $LTQ_4$ with dilation 2 rooted at $1010$.

Proof. We can embed a complete binary tree rooted at $1000$ into $LTQ_4$ with dilation 1, see Figure 2 (a), every node of $CBT_4$ is mapping to a node in $LTQ_4$ and every edge of $CBT_4$ is mapping to an edge in $LTQ_4$. While another complete binary tree rooted at $1010$ can be embedded into $LTQ_4$ with dilation 2, see Figure 2 (b). In the second embedding, edge $(1010,0000)$ in the complete binary tree is mapping to a path $P: 1010\rightarrow 0010\rightarrow 0000$ of $LTQ_4$. Every node of $CBT_4$ is mapping to a node in $LTQ_4$ and every edge of $CBT_4$ except edge $(1010,0000)$ is mapping to an edge in $LTQ_4$. By the definition of dilation, the dilation of the second embedding is 2. 

Fig. 2: (a) $LTQ_4$ and $CBT_4$ with dilation 1. (b) $LTQ_4$ and $CBT_4$ with dilation 2.

Lemma 5. A complete binary tree with $2^5 - 1$ nodes can be embedded into $LTQ_5$ with dilation 2, whose root is $00010$.

Proof. By the proof of Lemma 4, we have two complete binary trees which can be embedded into $LTQ_4$, we can embedded $CBT_5$ into $LTQ_5$ with dilation 2, the proof process is as follows. We prefix the label of each node of $CBT_4$ in the Figure 2 (a) with 0, and prefix the label of each node of $CBT_4$ in the Figure 2 (b) with 1, and then, we link the roots of this two complete binary trees with two edges $(00010,01000)$ and $(00010,11010)$. It upper steps can construct $CBT_5$. Edge $(00010,01000)$ of $CBT_5$ is mapping to the path $P_1: 00010\rightarrow 01010\rightarrow 01000$ of $LTQ_5$ and edge $(00010,11010)$ of $CBT_5$ is mapping to the path $P_2: 00010\rightarrow 10010\rightarrow 11010$ of $LTQ_5$, see Figure 3. By the definition of dilation, the dilation of the embedding is 2.

Every node of $CBT_5$ is mapping to a node in $LTQ_5$ and every edge of $CBT_5$ is mapping to an edge in $LTQ_5$, except edges $(1010,0000),(00010,01000)$ and $(00010,11010)$, while these three edges are mapping to a path of length 2 in $LTQ_5$, respectively. □

Fig. 3: $LTQ_5$ and $CBT_5$. 
Lemma 6. A complete binary tree with $2^6 - 1$ nodes can be embedded into $LTQ_6$ with dilation 2, whose root is 010010.

Proof. By the proof of Lemma 5, we have two complete binary trees which can be embedded into $LTQ_5$. We can embed $CBT_6$ into $LTQ_6$ with dilation 2, the construction process is as follows. To show the embedded tree clearly, Figure 4 omits some edges in $LTQ_6$. We prefix the label of each node of $CBT_5$ in the Figure 3 with 0, and prefix the label of each node of $CBT_5$ in the Figure 3 with 1, and then, we link the roots of this two complete binary trees with two edges (000010,010010) and (100010,010010). The upper step can construct $CBT_6$. Edge (000010,010010) of $CBT_6$ is mapping to the edge (000010,010010) of $LTQ_6$ and edge (010010,100010) of $CBT_6$ is mapping to the path $P$: 010010 $\rightarrow$ 110010 $\rightarrow$ 100010 of $LTQ_6$, see Figure 5. By the definition of dilation, the dilation of the embedding is 2.

Lemma 7. For any integer $n \geq 6$, a complete binary tree with $2^n - 1$ nodes can be embedded into $LTQ_n$ with dilation 2, whose root is $01^{n-5}0010$.

Proof. We prove this lemma by induction on the dimension $n$ of $LTQ_n$. According to Lemmas 6, this lemma holds when $n=6$. Supposing that the lemma holds for $n = \tau$ ($\tau \geq 6$), we will prove that the lemma holds for $n = \tau + 1$.

According to the induction of hypothesis, for any integer $\tau \geq 6$, for $LTQ_\tau$, a complete binary tree with $2^\tau - 1$ nodes can be embedded into $LTQ_\tau$ with dilation 2, whose root is $01^{\tau-5}0010$.

Now, we will prove that for any integer $\tau \geq 6$, a complete binary tree with $2^{\tau+1} - 1$ nodes can be embedded into $LTQ_{\tau+1}$ with dilation 2, whose root is $01^{\tau-4}0010$.

By the hypothesis, a complete binary tree $CBT_\tau$ with $2^\tau - 1$ nodes can be embedded into $LTQ_\tau$ with dilation 2, whose root is $001^{\tau-5}0010$. And a complete binary tree $CBT_{\tau+1}$ with $2^{\tau+1} - 1$ nodes can be embedded into $LTQ_{\tau+1}$ with dilation 2, whose root is $101^{\tau-5}0010$. 

\[\square\]
The complete binary tree $CBT_{r+1}$ with root $01^r-40010$ can be gained by $CBT^0_r$ and $CBT^1_r$ as figure 6. By the Definition 1, we have $(01^r-40010, 001^r-50010), (01^r-40010, 11^r-40010)$, and $(11^r-40010, 101^r-50010)$ are three edges of $LTQ_{r+1}$. The edge $a$ of $CBT_{r+1}$ is mapping to an edge of $LTQ_{r+1}$: $(01^r-40010, 001^r-50010)$. And the edge $b$ of $CBT_{r+1}$ is mapping to a path $P$ of $LTQ_{r+1}$: $01^r-40010 > 11^r-40010 > 101^r-50010$. A complete binary tree $CBT_{r+1}$ with root $01^r-40010$ can be embedded into $LTQ_{r+1}$. The dilations of the embedding $CBT^0_r$ and $CBT^1_r$ are 2, and $|P| = 2$, therefore, the dilation of embedding of $CBT_{r+1}$ into $LTQ_{r+1}$ is 2.

Fig. 6: The complete binary tree $CBT_{r+1}$.

**Theorem 1.** A complete binary tree $CBT_n$ with $2^n - 1$ nodes can be embedded into $LTQ_n$ with dilation 2.

**Proof.** By Lemma 1 and Lemmas 3 - 7, the theorem holds obviously.

Since the node number of $LTQ_n$ is $2^n$, and $CBT_n$ has $2^n - 1$ nodes, the expansion of this embedding is $\frac{2^n - 1}{2^n}$. When $n$ is big enough, the expansion is almost 1.

**Conclusions**

In this paper, the problem of embedding complete binary trees into locally twisted cubes is studied. We find the following result in this paper: for any integer $n \geq 2$, a complete binary tree with $2^n - 1$ nodes can be embedded into the $LTQ_n$ with dilation 2.

**Acknowledgment**

This work is supported by Natural Science Foundation of China (61170021, 61070169), Natural Science Foundation of Jiangsu Province under grant (SBK201220584), Specialized Research Fund for the Doctoral Program of Higher Education (20103201110018), Application foundation research of Suzhou of China (SYG201034), and sponsored by Qing Lan Project.

**References**


