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On some operator functional equations in locally convex algebras

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Abstract. This paper aims to prove the existence of the solutions of some operator functional equations of sinus type in universally bounded operator algebra, where the operator is defined on a locally convex space. Some results based on the work of B. Sz.-Nagy following the representation model of operator groups are established.

Preliminaries

A completely new area where applied research could be the solution to various unresolved issues from the engineering sciences, starting with the equations of motion of atomic particles up to complex signals theories is represented by operator functional equations within a larger framework: locally convex spaces.

This research is connected with the study of functional equations of sine and cosine in different algebraic-topological structures, as well as in the locally convex algebra of universally bounded operators defined on a locally convex space X, in connection with the theory of semigroups (groups) of operators. In this respect, there could be mentioned that a first step in this direction was made by S. Kurepa [14] obtaining a generalization of equations that characterize the trigonometric and hyperbolic functions by functional equations of matrices. World first works in operatorial trigonometric functional equations belong to S. Kurepa [14] and H.O. Fattorini [8].

In 1958, V. Ptak in [20] introduces the B− completeness of a locally convex space that allows an extension of the fundamental principles of functional analysis to such spaces. For such an extension, a result of D. Van Dulst [5] regarding the heredity property of a barreled space is needed.

Based on some known results obtained by Allan [1], F. G. Bonales and R. V. Mendoza [2], develop a version of spectral theory for bounded linear operators on barreled spaces.

All locally convex spaces will be assumed Hausdorff and over the complex field $\boldsymbol{C}$. A calibration for the locally convex space $\boldsymbol{X}$ is a family $\boldsymbol{P}$ of semi-norms generating its topology, in the sense that the topology of $\boldsymbol{X}$ is the coarsest with respect to which all the semi-norms in $\boldsymbol{P}$ are continuous. A calibration $\boldsymbol{P}$ is characterized by the property that the sets

$E(p,\epsilon)=\{x\epsilon X: p(x)<\epsilon\}, \epsilon>0, p\epsilon \boldsymbol{P}$

constitute a neighborhood sub-base at 0. (see [6, 7, 11])

We denote by $(\boldsymbol{X},\boldsymbol{P})$ a locally convex Hausdorff space with a calibration $\boldsymbol{P}$. A locally convex algebra is algebra with a locally convex topology in which the multiplication is separately continuous. Such an algebra is locally m-convex (l.m.c.) if it has a neighborhood base U at 0 such that $U \in$ U is convex, balanced ($\lambda U \subseteq U$ for $|\lambda| < 1$) and satisfies the semigroup property $U^2 \subseteq U$.

Any algebra with identity will be called unital. A unital l.m.c. algebra $\boldsymbol{A}$ is characterized by the existence of a calibration $\boldsymbol{P}$ such that each $p\in\boldsymbol{P}$ is submultiplicative $(p(xy)\leq p(x)p(y))$ and satisfies $p(e)=1$, where e is the unit element. A unital lmc algebra $\boldsymbol{A}$ is characterized by the existence of a calibration $\boldsymbol{P}$ such that each $p\epsilon\, \boldsymbol{P}$ is sub-multiplicative $(p(xy)\leq p(x)p(y))$ and satisfies $p(1)=1$. We recall that the dual $\boldsymbol{X}^*$ of a locally convex space $\boldsymbol{X}$ is endowed with the topology of uniform convergence on finite subsets of $\boldsymbol{X}$, denoted by $\sigma(\boldsymbol{X}^*, \boldsymbol{X})$. We also mention that U^0 denotes the polar of U ([11, 12 and 13]). According to V. Ptak [20], a locally convex space $\boldsymbol{X}$ is B-complete if a linear sub-space $Y\subset \boldsymbol{X}^*$ is $\sigma(\boldsymbol{X}^*, \boldsymbol{X})$ closed whenever $Y\cap U^0$ is closed for each 0-neighborhood

$U \subset \boldsymbol{X}$. Let $\boldsymbol{A}$ be a complex algebra of functions, like in [18]. By a locally multiplicative convex algebra (l.m.c.) we understand a topologically algebra whose topology is given by the family of sub-multiplicative semi-norms $\{p_\alpha\}$, i.e.

$p_\alpha(xy) \leq p_\alpha(x)\, p_\alpha(y)$, for any $x, y \in \boldsymbol{A}$ and $\alpha \in I$.

An algebra $\boldsymbol{A}$ is called proper if only the zero element annihilates the whole algebra $\boldsymbol{A}$, i.e. if $x\boldsymbol{A}=\boldsymbol{A}x=\{0\}$ then $x=0$. It is obvious that any semi-simple algebra is proper. Let $\boldsymbol{A}$ be a semi-simple commutative algebra. Then from [10] a multiplier is a mapping $T: \boldsymbol{A} \rightarrow \boldsymbol{A}$ which verifies

$Tx \cdot y = x \cdot Ty$, for all $x, y \in \boldsymbol{A}$.

We denote the set of all multipliers on $\boldsymbol{A}$ by $\boldsymbol{M(A)}$. We observe that the identity $x \cdot Ty = T(xy) = Tx \cdot y$ holds for some $x, y \in \boldsymbol{A}$ and thus the range TA and the kernel $ker\ T$ of T are both bilateral ideals of **A** (see [15, 17]).

By G. R. Allan [1], an element $x \in \boldsymbol{X}$, where $\boldsymbol{X}$ is a locally convex algebra, is said to be bounded in $\boldsymbol{X}$ if there exists $\alpha \in \boldsymbol{C}$ such that the set $\{(\alpha x)^n\}_{n\geq 1}$ is bounded in $\boldsymbol{X}$. The set of all bounded elements in $\boldsymbol{X}$ will be denoted by $\boldsymbol{X_0}$.

We call resolvent set in the Waelbroeck sense of an element x from a locally convex unital algebra $(\boldsymbol{X},\boldsymbol{P})$ the set of all elements $\lambda_0 \in \boldsymbol{C}_\infty(\boldsymbol{C} \cup \{\infty\})$ for which there exists $V \in V_{\lambda 0}$ such that the following conditions hold:

(a) the element $\lambda e - x$ is invertible in $\boldsymbol{X}$, for any $\lambda \in V \backslash \{\infty\}$;

(b) the set $\{(\lambda e - x)^{-1} : \lambda \in V \backslash \{\infty\}\}$ is bounded in $(\boldsymbol{X}, P)$.

The resolvent set will be denoted by $\rho(x)$. We also denote by $R(\lambda,x)=(\lambda e-x)^{-1}$ for $\lambda e - x$ invertible in $\boldsymbol{X}$. The function $\lambda \rightarrow R(\lambda,x)$ is called the resolvent function of x.

Introductory notions

Functional equations, both classic and operator ones were born from mathematical physics or from functional characterization of elementary functions.

D'Alembert in 1769 obtains cosine equation with 3 unknown functions in 1804, starting with composition of forces, Poisson reaches to the cosine functional equation

$$f(s+t)+f(s-t)=2f(s)f(t) \tag{2.1}$$

$$f(s+t)+f(s-t)=2f(s)g(t) \tag{2.2}$$

In 1909 D.R. Carmichael determines the analytic solutions of sine functional equation

$$f(s+t)f(s-t)=f^2(s)-f^2(t).$$

In the following we will present sine and cosine functional equations in a different algebraic-topological structure, namely in an l.m.c. algebra. First papers on the subject of trigonometric operator functional equations worldwide belong to S. Kurepa [14] and H.O. Fattorini [8].

It is known that if X is an arbitrary locally convex space, then the function $f{:}\boldsymbol{R} \longrightarrow X$ is measurable if there are a number of simple functions that converges a.e. on $\boldsymbol{R}$ to $f(t)$. The function f is strongly continuous if the condition $t \longrightarrow t_0$ implies $f(t) \longrightarrow f(t_0)$ in the topology given by the norm of X. The function $g{:}\ \boldsymbol{R} \longrightarrow B(X)$ is called operator function. It is continuous: if $t \longrightarrow t_0$ implies $g(t) \longrightarrow g(t_0)$ in the topology given by the norm of $B(X)$ and strongly continuous if for any $x \in X$, the function $f(t)=g(t)x$ is continuous.

Let $D(f)$ *and* $R(f)$ be the definition domain and respectively the co-domain of f. We say that the operator function f, $f{:}\ \boldsymbol{R} \longrightarrow B(X)$ is trigonometric if it satisfies a classical functional equation of a trigonometric function on $D(f)=\mathbf{R}$ with values in $R(f)=B(X)$. If $f{:}\boldsymbol{R} \longrightarrow B$, B being an arbitrary Banach algebra, J.A. Baker, generalizing S. Kurepa's results showed that if the function $f: \boldsymbol{R} \longrightarrow B$ verifies the cosine functional equation, then $f(0)=j$ is idempotent and there exists $b, c \in B$ such that $jb = bj = b$, $cj = c$.

If X is a complex Banach space and f is an operatorial strongly measurable function which satisfies $f(0) = I$ (I being the identity operator), then f is called cosine operatorial function. With their study dealt M. Sova [21], H.O. Fattorini [8], G. Da Prato [4], E. Giusti [3], B. Nagy [19], etc. Thus H.O. Fattorini showed that cosine and sine operatorial functions are closely related to the solutions of abstract differential equation

$u''(t) = \boldsymbol{A}u(t),\ t\epsilon\mathbf{R},\ u{:}\mathbf{R} \longrightarrow X,$

$\boldsymbol{A}$ being a linear operator, the Cauchy problem for the equation above is correctly placed. Also M.Sova [21], H.O.Fattorini [8], G.Da Prato [4], E.Giusti [3] and M.Watanabe [23, 24] independently obtained an analogue of Hille-Yosida's theorem of generating operatorial cosine functions. In the classical case ($f{:}\boldsymbol{R}\longrightarrow\boldsymbol{R}$), E.B. van Vleck [22] studied the functional equation of sinus

$$f(s+t+k)-f(s-t+k)=2f(s)f(t),\ k\epsilon\mathbf{R} \text{ — constant.} \tag{2.3}$$

If $f{:}\boldsymbol{R}\longrightarrow B(X)$ is a strongly measurable function that satisfies the above equation, then we say that f is an operator sine function of class $S(k)$.

Main results

Proposition 2.1 Let $f{:}D(f)\rightarrow X$ be a non-negative solution of *(2.1)*, where $D(f) = (G, \cdot)$ is a multiplicative group verifying the condition:

$$f(r\cdot s\cdot t)=f(r\cdot t\cdot s), \text{ for every } r,s,t\epsilon G \tag{2.4}$$

Then the following properties are true:

1) $f(s^2)+I_f=2f^2(s)$

2) $f(s^2)+f(t^2)=2f(s\cdot t)f(s\cdot t^{-1})$

3) $[f(s\cdot t)+f(s\cdot t^{-1})]^2=4[f(s^2)-I_f][f(t^2)-I_f]$

4) $[f(s\cdot t)-f(s)f(t)]^2=[f(s^2)-I_f][f(t^2)-I_f]$,

5) The closed sub-algebra $X_f = \{f(s){:}s\in G\}$ generated by the image of f is commutative and with the unit $f(e)= I_f$, I_f being the unit element from X_f.

Proof: For $s=t$ from (2.1) one can obtain 1). Replacing s by $s\cdot t$ and t by $s\cdot t^{-1}$ in (2.1) and by using "inferior commutativity" (2.4), one can deduce 2).

From 1), 2) and hypothesis (2.4) it results 3). By replacing

$$f(s\cdot t^{-1})=2f(s)f(t)-f(s\cdot t)$$

in 3), it results 4). To prove B_f is commutative one can inter-change s with t and we get

$f(t\cdot s)+f(t\cdot s^{-1})=2f(t)f(s)$, for all $t,s\epsilon G$.

From this and from (2.1), based on conditions (2.4) it follows

$f(s)f(t)=f(t)f(s)$ for every $s,t\epsilon G$,

equality which shows that sub-algebra B_f is commutative.

By taking $t=e$ in (2.1) we get $f(s)=f(s)f(e)$. Because the sub-algebra B_f is commutative it will result $f(e)=I_f$ is the unit element from B_f.

In the following we will prove that if $h{:}\ (G, \cdot)\rightarrow B$ is a homeo-morphism of the multiplicative group $(G, \cdot)$ on a multiplicative sub-group of B: $h(s\cdot t)=h(s)h(t)$ with the condition $h(s\cdot t) =h(t\cdot s)$ for every $s,t\epsilon G$ then the function $f{:}\ (G, \cdot)\rightarrow B$ defined by

$$f(s)=[h(s)+h(s^{-1})]/2 \tag{2.5}$$

is a solution of (2.1). So, the following two representation theorems of the cosine functional equation solutions hold.

Theorem 2.2 If $h{:}\ (G, \cdot) \rightarrow B$ is a homomorphism of the group $(G, \cdot)$ on a multiplicative sub-group of X and if $h(s \cdot t) = h(t \cdot s)$, for all $s, t \in G$ then the function defined by (2.5) is a solution of (2.1).

Proof: We have to show that the function f given by (2.5) with the help of homo-morphism h, verifies equation (2.1). This can be deduced from the sequence:

$$\begin{aligned}2f(s)f(t)&=2^{-1}[h(s)h(t)+h(s)h(t^{-1})+h(s^{-1})h(t)+ h(s^{-1})\, h(t^{-1})]\\ &=2^{-1}[h(s\cdot t)+h(s\cdot t^{-1})+h(s^{-1}\cdot t)+ h(s^{-1}\cdot t^{-1})]\\ &=2^{-1}[h(s\cdot t)+h(s\cdot t^{-1})]+ 2^{-1}[h(s\cdot t^{-1})+ h(s\cdot t^{-1})^{-1}]\\ &= f(s\cdot t)+f(s\cdot t^{-1}).\end{aligned}$$

Theorem 2.3 If $f:(G,\cdot)\to B$, where $(G,\cdot)$ is a multiplicative group and B a l.m.c. algebra is a solution of (2.1) with the properties:

i) f satisfies the condition (2.4)

ii) $f(e)=I_f\neq 0$

iii) it exists a Jordan curve (γ) which unites the points ±1 with one point $s_0 \in G$ such that $(\gamma) \cap \sigma(f(s_0))=\emptyset$, where $\sigma(f(s_0))$ is the Waelbroeck spectrum of $f(s_0)\in B$, then it exists a homomorphism of the group $(G,\cdot)$ on the locally convex algebra B for which the representation (2.5) exists.

Proof: The closed sub-algebra B_f (from Proposition 2.1) being commutative, according to condition iii), it exists an element $b\epsilon B_f$ such that

$$b^2=\alpha^2-I_f, \tag{2.6}$$

where $\alpha=f(s_0)\epsilon B_f$. Another consequence of condition iii) is the fact that $b\epsilon B_f$ is invertible. The homomorphism h is defined by

$$h(s)=f(s)+b^{-1}[f(s\cdot s_0)-f(s_0)f(s)]=b^{-1}[(b-\alpha)f(s)+f(s\cdot s_0)] \tag{2.7}$$

Then, from (2.1), (2.6) and (2.7) it results $h(s)h(s^{-1})=I_f$ or, equivalently $h(s)=h(s^{-1})$, for all $s\epsilon G$, leading to:

$$f(s)=2^{-1}[h(s)+h(s^{-1})]$$

for all $s\epsilon G$, where $h(s^{-1})$ is the inverse of $h(s)$ in the closed sub-algebra B_f meaning $h(s)\neq 0$ for every $s\epsilon G$.

It remains to prove h is a homo-morphism of G. To do that, from definition of h (2.7) we have:

$$h(s)h(t)=b^2\{f(s\cdot s_0)f(t\cdot s_0)+(b-\alpha)[f(s\cdot s_0)f(t)+f(t\cdot s_0)f(s)]+(b-\alpha)^2 f(s)f(t)\}.$$

Then, it can be deduced

$$2[f(s\cdot s_0)f(t)+f(t\cdot s_0)f(s)]=f(s\cdot s_0\cdot t)+f(s\cdot s_0\cdot t^{-1})+f(t\cdot s_0\cdot s)+f(t\cdot s_0\cdot s^{-1})=$$
$$=2f(s_0\cdot s\cdot t)+f(s_0(s^{-1}\cdot t)^{-1})+f(s_0 s^{-1}\cdot t)=2f(s_0\cdot s\cdot t)+2f(s_0)f(s^{-1}\cdot t)$$

and it follows

$$f(s\cdot s_0)f(t)+f(t\cdot s_0)f(s)=f(s\cdot s_0\cdot t)+\alpha[2f(s)f(t)-f(s\cdot t)], \tag{2.8}$$
$$2f(s\cdot s_0)f(t\cdot s_0)=f(s\cdot s_0\cdot t\cdot s_0)+f(s\cdot s_0\cdot(t\cdot s_0)^{-1})=$$
$$=f(s\cdot s_0\cdot s\cdot t)+f(s\cdot s_0\cdot s_0^{-1}\cdot t)=$$
$$=f(s_0\cdot s_0\cdot t\cdot s)+f(s\cdot t^{-1})=$$
$$=2f(s_0)f(s_0\cdot t\cdot s)-f(s_0\cdot(s_0\cdot t\cdot s)^{-1})+2f(s)f(t)-f(s\cdot t)=$$
$$=2f(s_0)f(s_0\cdot s\cdot t)-f((t\cdot t)^{-1})+2f(s)f(t)-f(s\cdot t).$$

With this, because of (2.4) we obtain:

$$2f(s\cdot s_0)f(t\cdot s_0)=2f(s_0)f(s_0\cdot s\cdot t)+2f(s)f(t)-2f(s\cdot t),$$

or

$$f(s\cdot s_0)f(t\cdot s_0)=f(s)f(t)-f(s\cdot t)+\alpha f(s_0\cdot s\cdot t). \tag{2.9}$$

From (2.6), (2.8) and (2.9) we have

$$h(s)h(t)=b^{-2}\{f(s)f(t)-f(s\cdot t)+\alpha f(s_0\cdot s\cdot t)+(b-\alpha)f(s_0\cdot s\cdot t)+\alpha[2f(s)f(t)-f(s\cdot t)]+(b-\alpha)^2 f(s)f(t)\}=$$
$$=b^{-2}[bf(s_0\cdot s\cdot t)+b(b-\alpha)f(s\cdot t)]=b^{-1}[f(s\cdot t)\cdot(b-\alpha)+f(s\cdot t\cdot s_0)]=h(s,t).$$

Going back, from the problem of determining all measurable functions $f: \mathbf{R}\to B$, where B is a l.m.c. algebra over $\mathbf{R}$ or $\mathbf{C}$, with the unit I_f, such that for all $s, t \in \mathbf{R}$, (2.1) is true with $f(e) = I_f$, one can obtain the result of the theorem:

Theorem 2.4 Let B be a l.m.c. algebra over $\mathbf{R}$ or $\mathbf{C}$ with the unit I_f, and let $f:\mathbf{R}\to B$ be a measurable function such that (2.1) is true with $f(e) = I_f$. Then it exists a unique element $a\in B$ such that

$$f(s)=\sum_{n=0}^{\infty}\frac{\alpha^n s^{2n}}{(2n)!}=\cosh(\sqrt{\alpha}\cdot s)$$

for all $s\in\mathbf{R}$, the series being absolutely convergent on $\mathbf{R}$.

Now, without the assumption that the l.m.c. algebra B has unit, the solution of (2.1) still can be determined.

Theorem 2.5 Let B be an l.m.c. algebra and $f: \boldsymbol{R}_{+} \to X$ be a measurable function such that (2.1) is true for $s > t > 0$. If it exists $\lim_{t\to 0_+} f(t) = j$, then $j^2=j$ (meaning j is idempotent) and there exist the elements $b, c \in B$ such that $bj=jb=b, cj=c, jc=\theta$.

Also f admits the representation:

$$f(s) = (j + \frac{s^2 b}{2!} + \frac{s^4 b^2}{4!} + \cdots) + c(sj + \frac{s^3 b}{3!} + \frac{s^5 b^2}{5!} + \cdots)$$

for all $s \in \boldsymbol{R}_+$.

Conversely with elements like j, b, c if f is defined as above, then it satisfies (2.1) for all $s, t \in \boldsymbol{R}$.

Similar representations for the functional equation of cosine (2.1) are also true for the functional equation of the cosine with two variables (2.2) satisfying continuity and measurability conditions for the functions f and g. The obtained results are based on the following lemma:

Lemma 2.6 If functions $f, g: \boldsymbol{R} \to B$ are satisfying the equation (2.2) and are strongly measurable, then f is strongly continuous and if

$$H=\{f(s), s \in \boldsymbol{R}: f(s)=\theta\}=\{\theta\},$$

then g is strongly continuous, even and satisfies the functional equation (2.1).

Theorem 2.7 If functions $f, g: \boldsymbol{R} \to B$ are strongly continuous, verifying the functional equation (2.2) and g satisfies the functional equation (2.1) and $g(0)=j \in B$, then j is idempotent and it exists the elements $a, b, c, d \in X$ with the proprieties $\alpha=\alpha j$, $b=bj=jb$, $c=cj$, $jc=\theta$, $d=dj$, such that we have the following representations:

$$f(s) = \alpha\,(j + b\frac{s^2}{2!} + b^2\frac{s^4}{4!} + \cdots) + d\,(js + b\frac{s^3}{3!} + b^2\frac{s^5}{5!} + \cdots), \quad (2.10)$$

$$g(s) = (j + b\frac{s^2}{2!} + b^2\frac{s^4}{4!} + \cdots) + c(jc + b\frac{s^3}{3!} + b^2\frac{s^5}{5!} + \cdots), \quad (2.11)$$

Conversely, with the elements $j, a, b, c, d \in B$, the functions f and g defined by (2.10) and (2.11) are verifying the functional equations (2.1) and (2.2).

We have studied the functional equation of sine (2.3) with the hypothesis $D(f)=\boldsymbol{R}$ and $R(f)=B$ (a l.m.c. algebra with the unit I). The element $f(r) \in B$, $r \in \boldsymbol{R}$ is a regular element of B that is for any $r \in B$, it exists $f(r)^{-1} \in B$.

Lemma 2.9 If B is a l.m.c. algebra, $f: \boldsymbol{R} \to B$ satisfies (2.3) and $f(r) \in B$, for any $r \in R$, is a regular element of B, then the following properties hold:

a) f is an odd function;

b) $g(s)=2^{-1}f(r)^{-1}[f(s+r)-f(s-r)]$, for all $s \in \boldsymbol{R}$

satisfies the functional equation (2.1), with $g(0) = I$ (I–unit element).

Theorem 2.10 Let B be a l.m.c. algebra with the unit I, $f: \boldsymbol{R} \to B$ a function satisfying (2.3), f strongly measurable on a subset $P \subset \boldsymbol{R}$ with positive Lebesgue measure and $f(r) \in B$, $r \in \boldsymbol{R}$, a regular element. Then it exists and they are unique the elements $b, d \in B$, with $bd=db$ such that

$$f(s) = d(Is + b\frac{s^3}{3!} + b^2\frac{s^5}{5!} + \cdots) \quad (2.12)$$

Conversely, if f has the form (2.12) and $bd=db$, then f satisfies the functional equation (2.3).

Applications

Depending on the number of parameters used in the natural description, these data can be classified as mono-dimensional (1D), where they depend on one parameter, or multidimensional (2D, 3D and nD), where the indexation depends on multiple parameters. 3D environment is illustrated by blocks of data field or 3D imagery such as geological, seismic, medical, radar, fluid dynamics, etc. Mathematically, these multi-dimensional data are generally treated as a function of one or more discrete or continuous variables - proper indexing parameters. The classes of models vary depending on the type of data and information considered relevant to these data as part of various applications.

In this context, as an example, expanding the Hilbert space L_2 to a locally convex sequentially complete space required in terms of types of data that must be processed, there is a need to characterize ergodic medium operators endowed with a certain type of limitation (see [16]).

The deterministic field half-plane (harmonic) is represented by a 2-D harmonic model for the 2-D, that is to say a finite sum of sinusoids. This model allows us to synthesize with very little parameters totally periodical textures as in figure 1 below. This type of textures is characterized in frequency domain by the presence of a finite set of points with high energies (peaks) and symmetrical with respect to the origin.

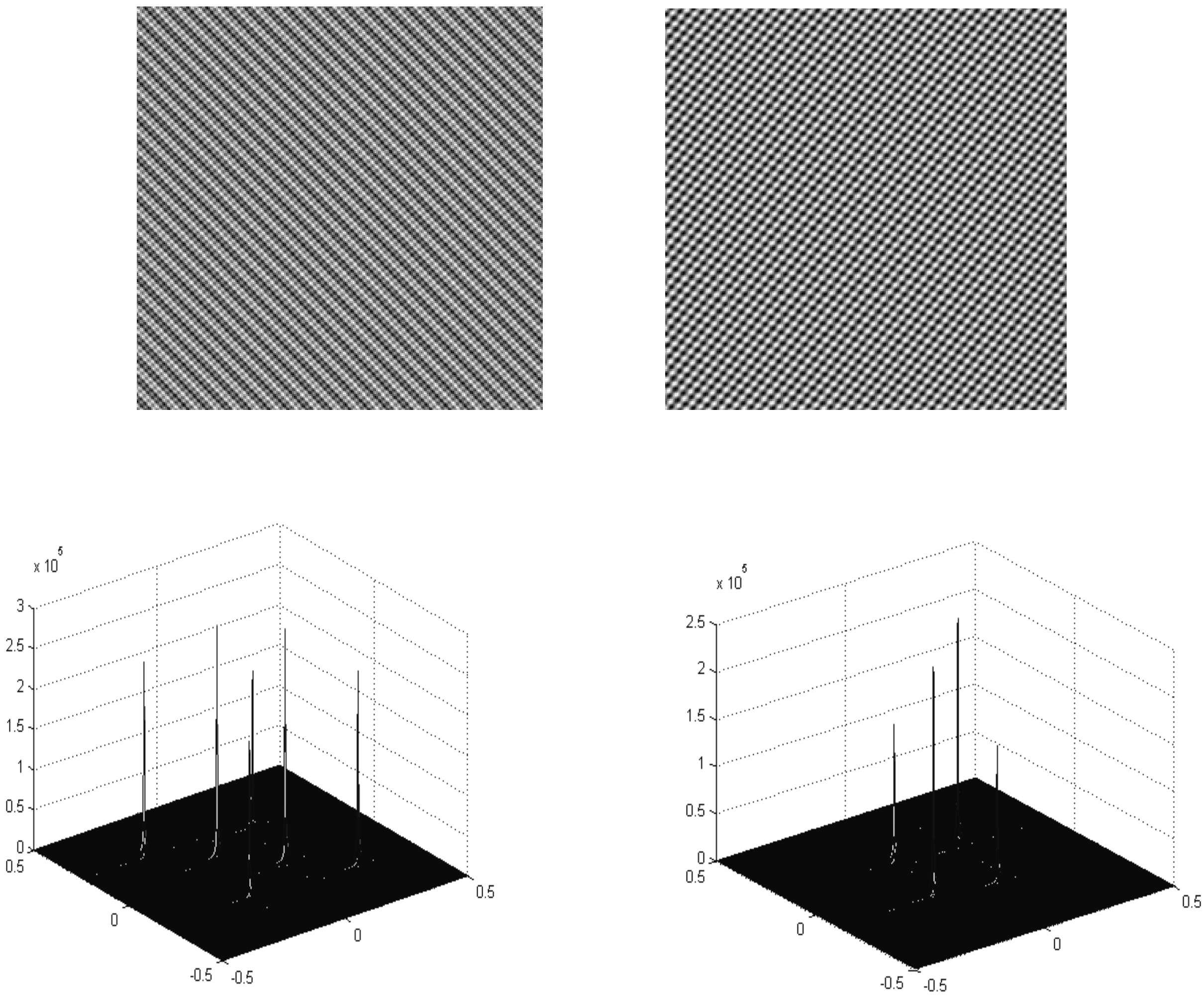

Figure 1. Examples of 2D harmonic textures. Spatial and frequency representation

Conclusions

Many areas of application today require treating a growing volume of digital data that appear important in a raw form or indexed by one or more parameters. These index data are subject to very general topics, including signal processing, field image or volume, with many applications in medicine and industry.

The studies carried out within this framework, show all types of functional equations developed on a broader category of spaces, namely the locally convex spaces and for a large class of operators acting on these spaces, such as universally bounded operators. Moreover, practical applications of these developments are numerous, especially due to the wide spread of possible generalizations, the problems occurring in real situations having to deal with a smaller set of constraints than ever before.

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