# An Innovative Own-Weight Cantilever Method for Measuring Young's Modulus in Flexible Thin Materials Based on Large Deflections

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**Abstract.** This report deals with an innovative method (*Own-Weight Cantilever Method*) to measure Young's modulus of flexible thin materials. A newly developed method is based on the large deformation theory considering large deformation behaviors due to own-weight in flexible thin materials. Analytical solutions are derived by using Bessel Functions. By means of measuring the horizontal displacement or the vertical displacement at a free end of a cantilever, Young's modulus can be easily obtained for various flexible thin and long materials. Measurements were carried out on a piano wire. The results confirm that the new method is suitable for flexible thin wires.

## Introduction

In recent years, flexible thin materials with very high performance are widely used. Therefore, large deformation analyses of these flexible materials have attracted attention considerably in the design of mechanical springs, fabrics and various thin-walled structures (aerospace structures, ship, car, etc.). An investigation of large deformation behaviors occurring in flexible materials is necessary for evaluation of mechanical properties such as Young's modulus. Here, a new Young's modulus measuring method (*Own-Weight Cantilever Method*) is proposed. Analytical solutions are derived by using Bessel Functions. By using this method, Young's modulus of thin and long flexible materials can be easily obtained by just measuring the horizontal displacement or the vertical displacement at the free end of the cantilever.

In order to assess the applicability of the proposed method, several experiments were carried out using a piano wire. As a result, it becomes clear that the new method is suitable for flexible thin wires. Besides the Own-Weight Cantilever Method studied here, the *Cantilever Method* [1], the *Circular Ring Method* [2,3] for a flexible single-layered material have already been reported, based on the nonlinear large deformation theory. Moreover, the *Cantilever Method* [4,5] for a flexible multi-layered material have been developed, based on the nonlinear large deformation theory.

## Theory

For small deformations, there are some testing methods, e.g. three- or four-point bending test for evaluating mechanical properties of various materials. These conventional tests are commonly used to obtain Young's modulus because of their simplicity. However, since the conventional methods are based on the small deformation theory, these methods are inapplicable directly to large deformation problems. Therefore, as the deformations of specimen grow larger, a more exact analysis is required to obtain accurate results. From this point of view, a new testing method (*Own-Weight Cantilever Method*) is derived considering large deformation behaviors. The new method can be applied to various thin, long fiber materials (Glass fibers, Carbon fibers, Optical fibers, etc.) and thin sheet materials.

**Basic Equations [6,7]**

A typical illustration of deflections is given in Fig.1 for a cantilever subjected to own-weight $w$ where $w$ is the distributed load per unit length with a supporting angle $\theta_0$. The horizontal displacement is denoted by $x$, the vertical displacement by $y$, and $\theta$ is the deflection angle. Furthermore, the arc length is denoted by $s$, the radius of curvature by $R$ and the bending moment by $M$. The relationships between $R, M, s, x, y$ and $\theta$ are given by:

$$1/R = -d\theta/ds, \quad M/EI = -d\theta/ds, \quad dx = ds \cdot \cos\theta, \quad dy = ds \cdot \sin\theta. \tag{1}$$

where $E$ is Young's modulus and $I$, the second area moment.

Putting $V_0$ as the vertical reaction at the origin O, $M_0$ as the bending moment at the origin O and $X$ as the integral of $\overline{X}$, the bending moment $M$ applied at an arbitrary position Q $(x, y)$ is

$$M = V_0 \cdot x - M_0 - \int_0^s w(x - \overline{x})d\overline{s} = wL \cdot x - M_0 - w[x\overline{s} - X]_0^s$$
$$= wL \cdot x - M_0 - w\{xs - X(s) + X(0)\} \tag{2}$$

The basic equation is derived from Eqs.1 and 2 in the form of:

$$EI \cdot d^2\theta/ds^2 + w(L - s)\cos\theta = 0. \tag{3}$$

Introducing the following non-dimensional variables and transforming the variables,

$$\xi = x/L, \quad \eta = y/L, \quad \zeta = s/L, \quad \gamma = wL^3/(EI), \quad \beta = ML/(EI) \quad . \tag{4}$$

equation 3 reduces to Eq.5

$$d^2\theta/d\zeta^2 + \gamma(1 - \zeta)\cos\theta = 0. \tag{5}$$

Assuming the following relationships in Eq.5,

$$\psi = \theta - (\theta_A + \theta_0)/2 \qquad [-(\theta_A - \theta_0)/2 \leq \psi \leq (\theta_A - \theta_0)/2]. \tag{6}$$

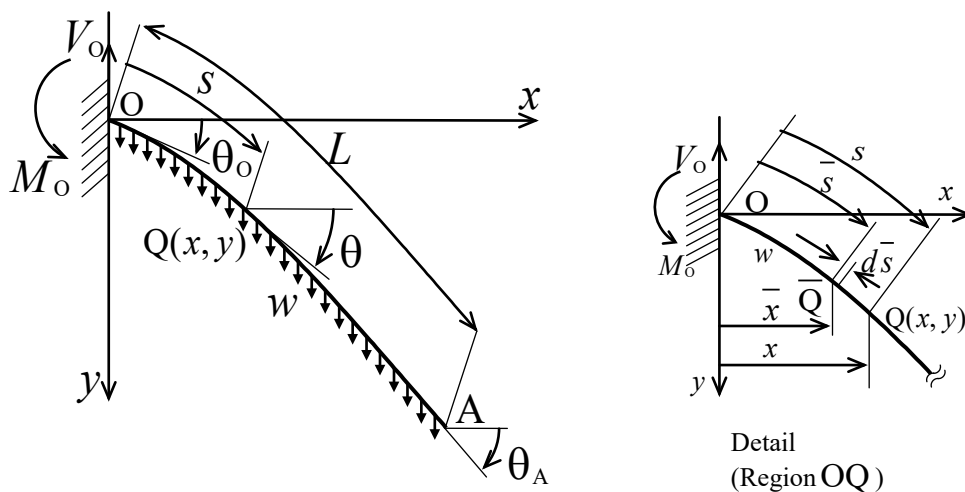finally, the nonlinear differential equation is obtained as follows.



Fig.1 Large deflections of a flexible cantilever subjected to own-weight.

$$\frac{d^2\psi}{d\zeta^2} - \frac{4J_1\{(\theta_A - \theta_o)/2\} \cdot \sin\{(\theta_A + \theta_o)/2\}}{\theta_A - \theta_o} \cdot \gamma \cdot \psi \cdot (1-\zeta)$$
$$= -J_0\{(\theta_A - \theta_o)/2\} \cdot \cos\{(\theta_A + \theta_o)/2\} \cdot \gamma \cdot (1-\zeta) \tag{7}$$

where
$$\sin\psi \cong 4J_1\{(\theta_A - \theta_o)/2\} \cdot \{\psi/(\theta_A - \theta_o)\}. \tag{8}$$

$$\cos\psi \cong J_0\{(\theta_A - \theta_o)/2\}. \tag{9}$$

The functions $J_n(x)$ appearing in Eqs.7-9 are Bessel Functions.

Considering the conditions $d\psi/d\delta|_{\delta=0} = 0 \; (d\theta/ds|_{\zeta=1} = 0)$ and $\psi|_{\delta=0} = (\theta_A - \theta_o)/2 \; (\theta|_{\zeta=1} = \theta_A)$, the non-dimensional maximum horizontal displacement $\xi_A(=\xi_{max}=x_A/L)$ ($x_A$:horizontal displacement at the free end A) and the non-dimensional maximum vertical displacement $\eta_A(=\eta_{max}=y_A/L)$ ($y_A$: vertical displacement at the free end A) are obtained as follows.

$$\xi_A = \int_0^{\zeta_A} \cos\theta d\zeta$$
$$= \int_0^{\zeta_A} \cos\left[\left\{\frac{\theta_A - \theta_o}{2} - q\right\} \cdot \left\{1 + \frac{m^2 \cdot \delta^3}{2 \cdot (4/3)} + \right.\right.$$
$$\left.\left. \frac{m^4 \cdot \delta^6}{2 \cdot 4 \cdot (8/3) \cdot (10/3)} + \cdots\right\} + q + \frac{\theta_A + \theta_o}{2}\right]d\zeta \tag{10}$$

$$\eta_A = \int_0^{\zeta_A} \sin\theta d\zeta$$
$$= \int_0^{\zeta_A} \sin\left[\left\{\frac{\theta_A - \theta_o}{2} - q\right\} \cdot \left\{1 + \frac{m^2 \cdot \delta^3}{2 \cdot (4/3)} + \right.\right.$$
$$\left.\left. \frac{m^4 \cdot \delta^6}{2 \cdot 4 \cdot (8/3) \cdot (10/3)} + \cdots\right\} + q + \frac{\theta_A + \theta_o}{2}\right]d\zeta \tag{11}$$

where
$$\delta = \gamma \cdot (1-\zeta) \qquad [0 \le \delta \le \gamma]. \tag{12}$$

$$q = \frac{(\theta_A - \theta_o) \cdot J_0\{(\theta_A - \theta_o)/2\} \cdot \cos\{(\theta_A + \theta_o)/2\}}{4J_1\{(\theta_A - \theta_o)/2\} \cdot \sin\{(\theta_A + \theta_o)/2\}}. \tag{13}$$

$$m^2 = \frac{16J_1\{(\theta_A - \theta_o)/2\} \cdot \cos\{(\theta_A + \theta_o)/2\}}{9\gamma^2 \cdot (\theta_A - \theta_o)}. \tag{14}$$

and the non-dimensional distributed load $\gamma$ is calculated from Eq.15.

$$-\frac{\theta_A - \theta_o}{2} = \left\{ \frac{\theta_A - \theta_o}{2} - q \right\} \times$$

$$\left\{ 1 + \frac{m^2 \cdot \gamma^3}{2 \cdot (4/3)} + \frac{m^4 \cdot \gamma^6}{2 \cdot 4 \cdot (8/3) \cdot (10/3)} + \cdots \right\} + q \quad . \tag{15}$$

Equations 10, 11 and 15 are the fundamental formulae to obtain the Young's modulus of the own-weight cantilever with an arbitrary supporting angle $\theta_0$, based on the nonlinear large deformation theory. From the viewpoint of the experiment, however, it is very simple and convenient to support a horizontal cantilever as a support technique of beams. Therefore, considering the availability of the new method, Young's modulus measurement at the supporting angle of $\theta_0$=0 (that is, horizontal) is described in the following section as a special case of "*Own-Weight Cantilever Method*"

**Measuring Techniques (Horizontal cantilever)**

As known from Eqs.10, 11 and 15, the formulae for measuring Young's modulus, based on the nonlinear large deformation theory are complicated in general. Therefore, for the sake of simplicity, the usage of the chart is recommended here by the author.

Although there are various methods in order to measure Young's modulus, two representative methods are introduced in this paper. The original data is a horizontal displacement $x_A$ or a vertical displacement $y_A$. Two charts (Nomographs) of $\gamma$ - $x_A$ relation (see Fig.2) and $\gamma$ -$y_A$ relation (see Fig.3) are presented, which were computed previously by using Eqs.10, 11 and 15. The calculation is repeated until the assumed angle $\theta_A$ at the free end coincides with the calculated angle $\theta_A$ based on Eq.(15). When the two values agree with each other, the deformed shape is determined ultimately. In Figs.2 and 3 (symbol ●), various sketches represent deformed beams on the specific conditions $x_A$, $y_A$.

The following formula based on Eq.4 is utilized, when the chart is used for calculating Young's modulus.

$$E = wL^3 / (\gamma I). \tag{16}$$

**Method 1: Measurement of $x_A$**

The usage of this method is shown below. A chart (Nomograph) is given in Fig.2, illustrating the relationship of $\gamma$ and $x_A/L$ ($\xi_A$). Using this chart, Young's modulus $E$ in a cantilever can be calculated from the relational expression given in Eq.16. As an example, Young's modulus $E$ is obtained for a piano wire (SWP-A) with diameter: $d$=0.9 mm, distributed load per unit length: $w$=49.3×10$^{-3}$ N/m.
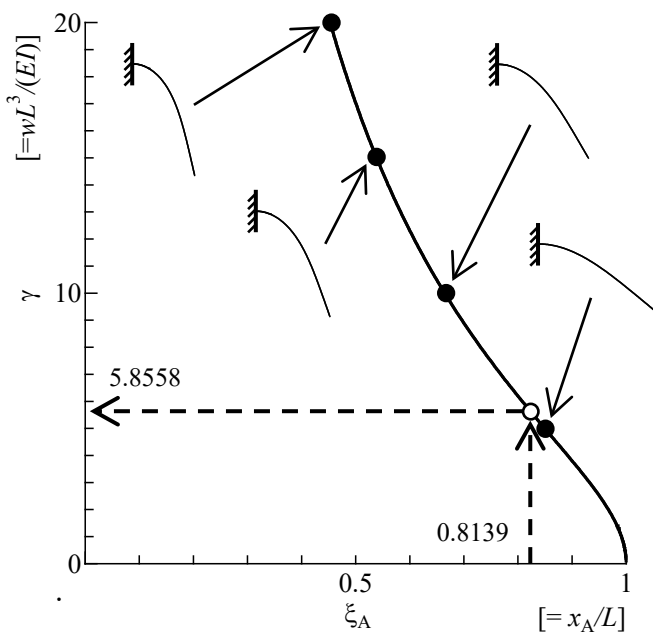
Fig.2 Non-dimensional chart for the finding the parameter $\gamma$ when the horizontal displacement $x_A$ is given.
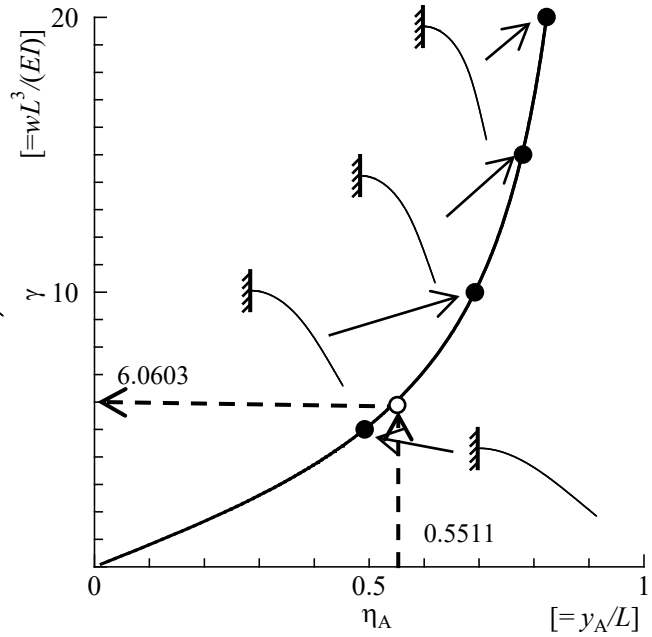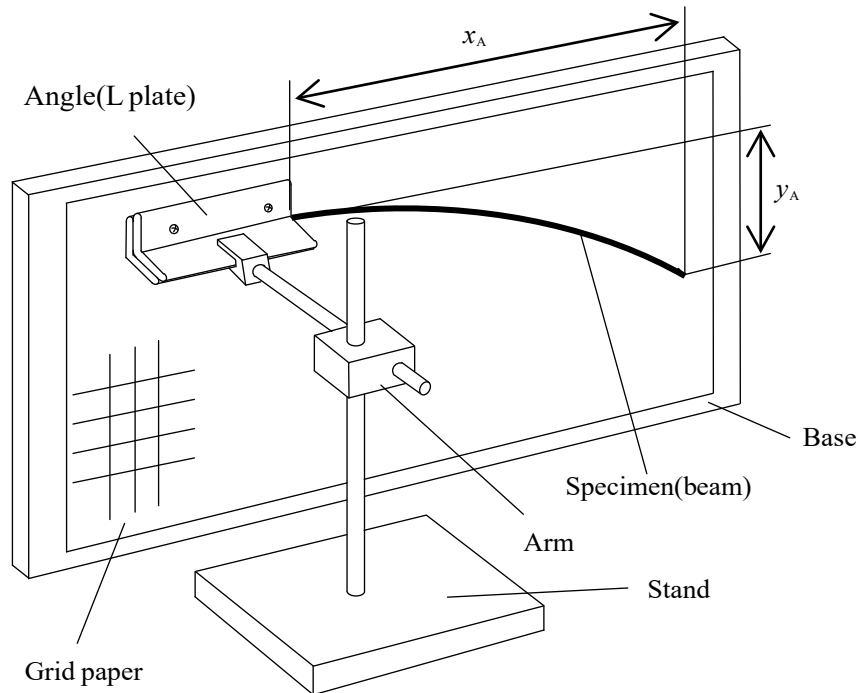
Fig.3 Non-dimensional chart for the finding the parameter $\gamma$ when the vertical displacement $y_A$ is given.

Fig.4 Experimental set-up

When $L$=900.0 mm, $x_A$ =732.5 mm (i.e., $\xi_A$= $x_A/L$ =0.8139) is measured and then $\gamma$ is taken from Fig.2 ($\gamma$=5.8558). Therefore, Young's modulus $E$ is calculated from Eq.16 as follows.

$$E = \frac{wL^3}{\gamma I} = \frac{49.3 \times 10^{-3} \times (0.9)^3}{5.8558 \times 3.221 \times 10^{-14}} \doteqdot 190.3 \times 10^9 \ \text{N/m}^2 = 190.3 \ \text{GPa}\ .$$

**Method 2: Measurement of $y_A$**

A similar chart (Nomograph) is given in Fig.3, illustrating the relationship of $\gamma$ and $y_A/L$ ($\eta_A$). Using this chart, Young's modulus $E$ can be calculated from Eq.16. As an example, Young's modulus $E$ is obtained for a piano wire (SWP-A) mentioned above. When $L$=900.0 mm, $y_A$ =496.0 mm (i.e., $\eta_A$= $y_A/L$ =0.5511) is measured and then $\gamma$ is taken from Fig.3 ($\gamma$=6.0603). Therefore, from Eq.16, Young's modulus $E$ is calculated as follows.

$$E = \frac{wL^3}{\gamma I} = \frac{49.3 \times 10^{-3} \times (0.9)^3}{6.0603 \times 3.221 \times 10^{-14}} \fallingdotseq 183.9 \times 10^9 \ \text{N}/\text{m}^2 = 183.9 \ \text{GPa}.$$

**Experimental Investigation**

In order to assess the applicability of the proposed method, several experiments were carried out using a thin piano wire (SWP-A) with diameter: $d$=0.38 mm, distributed load per unit length: $w$=8.78×10$^{-3}$ N/m. An experimental set-up is shown in Fig.4. Young's modulus of SWP-A by applying the Method 1 and Method 2 are shown in Figs. 5 and 6, respectively. In the experiment, a horizontal displacement $x_A$ and a vertical displacement $y_A$ at the free end are measured by using a grid paper with 1mm spacing. Here, the influence of the length upon Young's modulus was examined for several lengths of a cantilever.

The measured values of Method 1 and 2 remain nearly constant for various lengths $L$ in the range of 350.0-700.0 mm and the standard deviation (S.D) is small. That means Young's modulus shows a
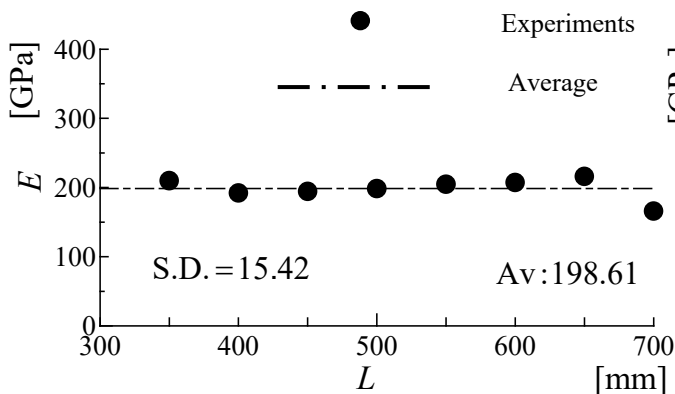


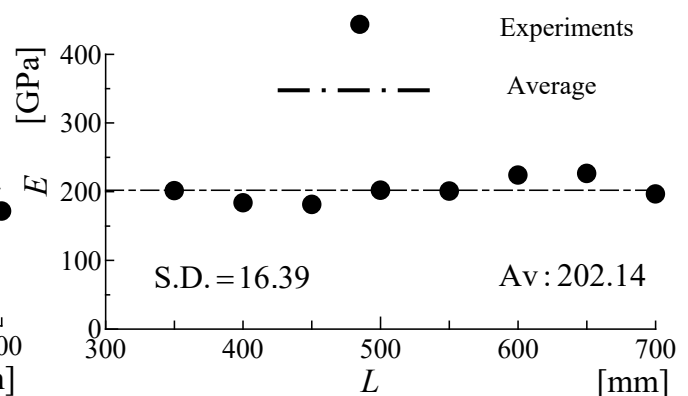Fig.5 Young's moduli for $\phi$=0.38mm piano wires (SWP-A)(from Data $x_A$).

Fig.6 Young's moduli for $\phi$=0.38mm piano wires (SWP-A)(from Data $y_A$).

stable value considerably within the experimental range of the slenderness ratio $L/d$≈900-1800. As a whole, the mean Young's moduli determined by the two methods are reasonably in good agreement with each other although Method 1 or 2 has a little scatter in the values. Therefore, there is little reason to choose between Method 1and Method 2 to measure Young's modulus. As a reference, the value measured by using conventional three-point bending test based on the small deformation theory is $E_0$=205.8 GPa.

**Conclusions**

The Own-Weight Cantilever Method is developed as a new and simpler material testing method for measuring Young's modulus in a flexible thin material.

The principal conclusions are drawn as follows from the results of theoretical and experimental analyses.

(1) The new method is based on the nonlinear large deformation theory.
(2) Two representative charts are drawn on the basis of the proposed theory for the sake of simplicity.
(3) Based on the new idea, a set of testing devices was designed.
(4) A thin piano wire (SWP-A) was tested. Experimental results clarify that the new method is suitable for measuring Young's modulus in a flexible thin material.
(5) Based on the assessments, the proposed method is applicable widely to Young's modulus measurement in thin sheets and fiber materials (e.g., steel belts, glass fibers, carbon fibers, optical fibers, etc.).

**References**

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