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Game-Theoretical Viewpoint to Optimal Design

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Abstract: The main goal of this study is to improve the game-theoretical approach for making the best structural design choices in situations where we are not sure what the external loads will be. The game formulation is used to improve games with different control functions. The games on a unit square need at least two levels because of the basic need for optimization. In a stratified game, the levels are called the substratum and superstratum levels. At the basic level, the way to improve something is like the payoff function, with "ordinal players" trying to make the respective payoff functions as small or as big as possible. The strategies used by these "ordinal players" are limited by their resources, and the value of the substratum game on the unit square is reflected in these strategies. At the upper level of the game, its value depends on the design parameters. The "designer," or the main player, controls these design parameters. If there are multiple players with different goals, the game decides how to handle that. On the other hand, if there is only one cardinal player, finding the maximum value of the superstratum game is the same as a regular optimization. The following text will give you all the information you need. The text presents the exact solutions for two different types of quadratic game formulations: the matrix game and the game on the unit square.

1. Principles of Interdependent Decision Making

Game theory is a field of mathematics that examines mathematical models of decision-making in situations where there are two or more parties involved in a competitive endeavor to exert influence over the progression of the conflict in a manner that aligns with their individual objectives. The way we think about making the best decisions using math is called operations research. This means that game theory is a type of math that is used in operations research. Mathematical game theory is an important part of operations research. The subject has been used in many different areas, such as economics, management, industry, agriculture, the military, construction, trade, transportation, communications, and more.

Game theory is a way of thinking about situations where people are both competing and working together. It helps us understand what happens when people make decisions that are based on reason. This theory is based on the principles of mainstream decision theory and economics. According to these principles, people behave rationally when choosing actions that will give them the best results, given the limits they face. Game theory is the study of how people make decisions when the result depends on what other people do. A key point to know about these situations is that each person in charge has only some control over the final results. Game theory is the scientific study of how people make decisions when they are playing a game. The goal of game theory is to understand how people will interact with each other in a way that leads to a specific result. But in many cases, one of the people involved can't be considered a conscious individual with preferences and goals. Therefore, it's not fair for the other players to assume that this person will act logically.

The way operations research tasks are grouped depends on how much information the person making decisions has about the situation being studied. The most basic information about the situation is either deterministic or stochastic. In the deterministic case, the conditions under which decisions are made are fully known. On the other hand, the stochastic case requires a thorough understanding of the possible conditions and their associated probability distributions. In these cases, the goal is to find the highest or lowest possible value of a function (or its mathematical expectation) while considering the given limits. In game theory, uncertainty can come from many different sources. However, it is commonly understood that these events are often the result of deliberate actions by the other parties involved, who are motivated by their own objectives. In this regard, game theory is often seen as the theoretical framework that includes mathematical models of making the best decisions in situations where there is conflict. Therefore, it can be suggested that game theory models can explain many different things. These include things like economic, legal, and class conflicts; human interaction with nature; and biological struggles for survival. In the study of game theory, all of these models are called "games."

Game theory was utilized in the formulation of optimal solutions [1], [2], [3]. The work [4] is devoted to game control, observation and search problems for dynamical systems. The systems we're looking at are explained using equations or formulas that involve finite differences. We assume a conflict situation, which is a common element in game theory. The system is influenced by two different forces. One of these forces is controlled by one party, and the other force is not controlled by the controlling party. This other force is either associated with the opponent or with factors that are not known or controllable ("nature"). The idea of "playing against nature" was explained in detail in the standard book [5]. Problems are posed and solved about how to control the system in the presence of various kinds of adversary influences, uncertain factors, incompleteness, and imperfection of incoming information about the process. The best possible result is known as "optimality of control." This is understood in the minimax sense. That means it's understood from the point of view of the controlling party. They want to get the best result possible. They want this result without the other party's influences.

In order to ensure compliance, the role of the payoff function can be explained in terms of the smallest eigenvalue of the inverse operator (in the continuous case) or the response matrix (in the discrete case) [6]. This is a key part of the game, and it's called the upper game value. In a certain type of convex game, this specific value is equal to the game's value. The best strategies for the "designer" and "nature" are set [5]. The same payoff functional, which is a way of measuring success, is used for both players. This game is different from the classic matrix game. The "designer" changes the payoff matrix coefficients, while "nature" adjusts the left and right vectors of the payoff function. The left vector is the same as the transposed right vector.

The main goal of this study is to use a game-theoretical approach to make the design of structures better when it is not clear what the external conditions will be. [7]. The basic idea can be explained simply as follows:

In the context of classical structural optimization, the "cardinal" players are assumed to perform the role of "control functions." The "cardinal" players also change the rules of the game. In structural optimization, the "cardinal" players are responsible for determining the coefficients of the governing equations. Their job is to set the rules of engagement, which are the conditions under which some players fight each other. The "cardinal players" have predetermined actions because their decisions are made during the design phase. In the stratified game approach, the "cardinal" players are defined as the "super-stratum."

Also, the other participants act according to the governing equations. In the context of the stratified game approach, the "ordinal" players are the "substratum." The "ordinal" players are allowed to make decisions within their assigned level. However, they are not allowed to change the governing equations. The "original players" can change their actions because they make decisions during the exploration phase. The "ordinal" participants represent external forces. In the business world, external factors are often referred to as "nature." Because of this, the games are called "games against nature."

The "ordinal" participants try to counteract the effects of nature to mitigate potential risks or to achieve the most favorable outcome. People in these roles are now called "operators." The relationship between the two "ordinal" players, nature and operators, is studied using game theory. This is a field of study that uses common principles to analyze strategic interactions. The payout matrix is important for Player I and the profit that Player I makes if he uses his strategy when Nature is in a certain state. It may be better to think of Nature's strategies as "states" instead of strategies. The main idea of this study is that we don't know what causes nature to choose one state over another. Statistical decision theory is a way of analyzing situations where we have to make a decision. In these situations, probabilities (whether they are objective or subjective) are important factors. At the ordinal level, the focus will be on the analysis of zero-sum games in both matrix and unit-square formulations. The definitions in question were visible within the work [8]. The technique could be extended to the bi-matrix and bi-integral formulations.

The payoff matrix is modified by the player who is acting in accordance with the "cardinal" strategy. The change is made to help the player who wins. Once the "substratum" game has started, you can't change the payout matrix. The "cardinal" players have the same names as those used in classical structural optimization. The "cardinal" players' freedom is described as "control functions" or "design parameters." If there's only one "cardinal" player, there won't be any conflicts of interest. If there are several "cardinal" players with different interests, there might be a conflict. The conflict mentioned earlier leads to the "superstratum" game on the upper level. This is not an unusual situation in the engineering field. For example, the system designer's goal is to make the system work as well as possible. The production designer wants to use the system in the best and cheapest way. The operating ecology manager's job is to make sure that the system emits as little as possible while it is being used. The production ecology manager knows that it is important to reduce emissions during manufacturing. The customer's main goal is to reduce the system's overall cost and operational expenses. However, these goals are not always the same, which can lead to problems among the "main players."

This way of thinking can be used for other problems that involve making decisions at different levels. To illustrate, lawmakers can be considered the "super-stratum" of a social game. The actions of the superstratum players directly impact the game's governing equations, including those related to taxation and ecological laws. Meanwhile, the players in the "substratum" follow the laws that define their goals and the way they go about achieving them.

2. Antagonistic Matrix Stratified Games

2.1 Common matrix games

This section contains basic information about the theory of finite matrix games. Researchers have studied how to find the best mix of strategies in a group of mixed strategies, the properties of optimal mixed strategies, and methods for solving matrix games. To mathematically describe a game, you need to list all the players and their strategies. This description also includes the exact amounts of money that each player will win after the other players choose their strategies. Consequently, the game is transformed into a formally defined procedure, which lends itself to unique mathematical analysis.

Games can be grouped in different ways. First, non-coalition games in which each coalition (a set of players acting together) consists of only one player. The cooperative theory of non-coalition games is a way of thinking about how players can work together temporarily to form groups and share any winnings they win or make decisions together. Second, coalitional games are games where players are divided into teams according to the rules of the game. Coalition members can share information and make decisions together. In the present study, the focus is exclusively on pure conflict situations. The antagonistic games will be displayed for the sake of simplicity.

Game theory is a field of study that starts with a simple model called a "matrix game." In this model, two players compete, and each player has a limited set of strategies. When one player wins, the other player loses:

$$\Gamma_{K[\boldsymbol{a}]} = (X, Y, K[\boldsymbol{a}]), \tag{1}$$

where X and Y are nonempty sets, and the function $K[\boldsymbol{a}] : X \times Y \to R^1$ is called an antagonistic stratified game in normal form. The elements $x \in X$ and $y \in Y$ are called the strategies of ordinal players 1 and 2, respectively, in the game $\Gamma_{K[\boldsymbol{a}]}$. The elements of the Cartesian product $X \times Y$ (i.e., the pair of strategies (x, y) where $x \in X$ and $y \in Y$) are situations, and the function $K[\boldsymbol{a}]$ is the win function of player 1. The payoff of player 2 in situation (x, y) is assumed to be equal to

$[-K[\boldsymbol{a}](x,y)]$. Therefore, the function $K[\boldsymbol{a}]$ is also called the win function of game $\Gamma_{K[\boldsymbol{a}]}$ itself, and game $\Gamma_{K[\boldsymbol{a}]}$ is called a zero-sum game.
In the standard game theory [9] [10], the elements of the matrix are defined as fixed values. Both players know these values (i.e., "ordinal players"). The main difference between the standard game formulation and the actual formulation is that the matrix elements depend on the control of the "cardinal" player. There are two levels in the game. The "cardinal" player acts on the "upper level." The "ordinal" players perform on the "lower level."
Accordingly, using the accepted terminology to define game $\Gamma_{K[\boldsymbol{a}]}$, it is necessary to define the sets of strategies X, Y of ordinal players 1 and 2 and also the winning function $K[\boldsymbol{a}]$, defined on the set of all situations $X \times Y$.

2.2 Stratified matrix games

The stratified game $\Gamma_{K[\boldsymbol{a}]}$ is interpreted as follows. Ordinal players simultaneously and independently choose strategies $x \in X$, $y \in Y$, as explained in [8]. In the substratum game, ordinal player 1 then receives a payoff equal to $K[\boldsymbol{a}](x,y)$, and ordinal player 2 receives $(-K[\boldsymbol{a}](x,y))$. The elements $\boldsymbol{a} \in \mathrm{A}$ are called the strategies of cardinal players in the stratified game $\Gamma_{K[\boldsymbol{a}]}$.
In a stratified game, the elements $\boldsymbol{a} \in \mathrm{A}$ are referred to as the strategies of cardinal players. The superstratum game pertains to the strategies for cardinal players $\boldsymbol{a}$ that ensure the maximal and minimal values of the substratum payoff function. When there is only one cardinal player, the superstratum game becomes an optimization problem. On the other hand, when there are two or more cardinal players, their interests may be in opposition. This is known as an antagonistic game. The total payoff for both players on the lower level is zero. The goal function on the "upper level" will be the payoff on the "lower level" game of the oppositely acting ordinal players, e.g., the "nature" and "operator". The "designer" makes this goal function better by thinking about it.
In the matrix game, ordinal player 1 is assumed to have only m strategies. The set of strategies available to the first ordinal player, X, must be ordered. That is, a one-to-one correspondence must be established between X and $M = \{1, 2, \ldots, m\}$. The same process must be repeated for the second ordinal player, with $N = \{1, 2, \ldots, n\}$ and Y.
The substratum matrix game Γ is defined by the matrix $\boldsymbol{A} = \boldsymbol{A}[\boldsymbol{a}]$:

$$\boldsymbol{A} = \{a_{ij}\}, \tag{1}$$

$$\begin{gathered} a_{ij} = K(x_i, y_j) \\ (x_i, y_j) \in X \times Y, \quad (i,j) \in M \times N, \\ i \in M, \quad j \in N. \end{gathered} \tag{2}$$

The game Γ is realized as follows. Player 1 selects a row, $i \in M$, and player 2 (simultaneously with player 1 and independently of him) chooses a column, $j \in N$. Ordinal player 1 then receives a payoff, a_{ij}, and ordinal player 2 receives $-a_{ij}$. If the payoff is a negative number, the original player has actually lost money.
We denote the substratum game Γ with win matrix $\boldsymbol{A}[\boldsymbol{a}]$ by $\Gamma_{A[\boldsymbol{a}]}$ and refer to it as an $(m \times n)-$game, in accordance with the dimensions of matrix $A[\boldsymbol{a}]$ with the fixed values of strategies for cardinal players $\boldsymbol{a}$.

2.3 Equilibrium situations and minimax principle

The idea of optimality in the lower layer is derived from the construction of an equilibrium situation, known as the equilibrium principle. If there's a balanced situation, the minimax is the same as the maximin. According to the definition of the equilibrium situation, each player can communicate their optimal (maximin) strategy to the opponent, and neither player can obtain an additional benefit from it. In the antagonistic game, the situation is called an equilibrium situation or a saddle point. At the saddle point, a part of the matrix is the lowest value in its row and the highest value in its column. Any two optimal strategies are a balanced situation, and the reward in it is equal to the value of the game. Games where both players have a limited number of strategies are called substratum matrix games.

It makes sense that a situation (x^*, y^*) in the game $\Gamma_{K[\boldsymbol{a}]} = (X, Y, K[\boldsymbol{a}])$ is optimal if changing from it is not good for any of the players. When we have a situation like this (x^*, y^*), we call it an equilibrium. The idea that something should be built in a certain way to create an equilibrium situation is called the equilibrium principle.

Now, we will look at how the equilibrium principle and the minimax and maximax principles work together in an antagonistic game. For the substratum game $\Gamma_{K[\boldsymbol{a}]} = (X, Y, K[\boldsymbol{a}])$ to be in a balanced state, it is necessary and sufficient that there exist a minimax and a maximin:

$$\min_y \sup_x K[\boldsymbol{a}](x, y), \ \max_x \inf_y K[\boldsymbol{a}](x, y), \tag{3}$$

and the equality is satisfied:

$$v^*[\boldsymbol{a}] = \min_y \sup_x K[\boldsymbol{a}](x, y) = \max_x \inf_y K[\boldsymbol{a}](x, y) = v^{**}[\boldsymbol{a}]. \tag{4}$$

Equation 0 shows how the equilibrium principle and the minimax and maximin principles in an antagonistic game are connected. Games that have a balanced state are called well-defined games. Therefore, this theorem creates a way to define a "lower level" game. It can be rewritten like this: If there's a balanced situation, then minimax equals maximin. According to the definition of a balanced situation, each "ordinal" player on the "lower level" can communicate their optimal (maximin) strategy to the opponent. This means that neither player can gain an advantage.

2.4 Mixed solutions

Now, let's say that in the game, there's no balanced situation in the "lower level" game, called "game $\Gamma_{A[\boldsymbol{a}]}$. In this situation, the maximin and minimax strategies aren't the best. Also, it's not good for the players to follow them, because they can win more money. However, telling the other player about your strategy might lead to even bigger losses than if you had chosen the maximin or minimax strategy. In this case, it makes sense for "ordinal" players to act randomly. This provides the greatest secrecy in the strategy choice. The result of the choice cannot be known to the opponent because the random mechanism is not known to the player until it is realized. The random variable whose values are the "ordinal" player's strategies is called their mixed strategy. With that definition in mind, we'll call the first set of strategies "pure." A random variable is defined by its distribution. We will identify a mixed strategy with a probability distribution on the set of pure strategies. Therefore, the range of a mixed strategy includes pure strategies that are chosen with positive probabilities. The pair of mixed strategies of the "ordinal" players in a matrix game is $(\boldsymbol{x}, \boldsymbol{y})$.. Every matrix game has a mixed strategy equilibrium situation [11]. In simple terms, if a solution exists for mixed strategies, "ordinal" players can always remove the uncertainty of their strategy choice. They can do this by choosing strategies randomly from the set of pure strategies. This happens simultaneously and independently on the "lower level" game.

A random variable is defined by its distribution. We will identify a mixed strategy with a probability distribution on the set of pure strategies of ordinal players. Therefore, the strategy $\boldsymbol{x}$ of player 1 in the substratum game is a m-dimensional vector, and it is constrained by the following equation:

$$\boldsymbol{x} = (\xi_1, \dots, \xi_m), \quad \|\boldsymbol{x}\|_{p,m} = 1. \tag{5}$$

where the L^p norm in the real vector space $\mathbb{R}^q$ is defined as [12]:

$$\|\boldsymbol{x}\|_{p,q} \stackrel{\text{def}}{=} \left(\sum_{i=1}^{q} {\xi_i}^p\right)^{1/p}. \tag{6}$$

In the same way, player 2's mixed strategy, $\boldsymbol{y}$, is a n-dimensional vector:

$$\boldsymbol{y} = (\eta_1, \dots, \eta_n), \quad \|\boldsymbol{y}\|_{p,n} = 1. \tag{7}$$

The positive natural number p is what decides what kind of game it is. Remember, the numbers n and m can't be the same.

2.5 Linear matrix games with the probability distributions of the strategies of players

If $p = 1$, the values $\xi_i \geq 0$ and $\eta_i \geq 0$ are the probabilities of choosing pure strategies $i \in M$ and $j \in N$, respectively, when ordinal players use mixed strategies x and y. We will use X and Y to represent the sets of mixed strategies of the first and second players, respectively. It's clear that the set of mixed strategies for each player is a compact set in the corresponding finite-dimensional Euclidean space (a closed, bounded set). A mixed set is a type of strategy that adds to the options available to the player. An arbitrary matrix game is well-defined within the class of mixed random strategies. The von Neumann theorem of matrix games states that, when $p = 1$, every matrix game has a balanced situation within the context of mixed strategies.
Setting p = 1 is normal for the application of game theory in economics, political science, and the social sciences [9] [13].
This shows that the mixed strategy for ordinal players is just a probability distribution over their pure strategies. Any event must have a positive probability, and the total probability of all events must be one. Therefore, any mixed strategy must meet the following conditions:

$$\|\boldsymbol{x}\|_{1,m} = 1, \quad \|\boldsymbol{y}\|_{1,m} = 1. \tag{8}$$

The proof of the theorem is constructive. This means that it can be used to solve the problem. The proof reduces the solution of the matrix game for $p = 1$ to a linear programming problem. Therefore, any method used to solve linear programming problems can be used to solve matrix games. The simplex method is the most common way to solve these problems.
2.6 Quadratic matrix games with the restrictions on the vector norms of the strategies of players
If $p = 2$, the values ξ_i and η_i are the Euclidian coordinates of vector strategies $\boldsymbol{x}$, $\boldsymbol{y}$ of the ordinal players. The Euclidean length of a vector $\boldsymbol{x}, \boldsymbol{y}$ in the real vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are given by their Euclidean norms:

$$\|\boldsymbol{x}\|_{2,m} \equiv \left(\sum_{i=1}^{m} \xi_i{}^2\right)^{\frac{1}{2}}, \qquad \|\boldsymbol{y}\|_{2,n} \equiv \left(\sum_{i=1}^{n} \eta_i{}^2\right)^{1/2}. \tag{9}$$

As the examples show, this is a common situation in engineering and physical applications. In these situations, the actions available to the players are limited. The strategy vectors are limited to a value of one:

$$\|\boldsymbol{x}\|_{2,m} \leq 1, \quad \|\boldsymbol{y}\|_{2,n} \leq 1. \tag{10}$$

The payoff function of the game matrix on the lower substratum level is expressed as follows:

$$\mathcal{J}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}] = \frac{1}{2}\langle \boldsymbol{A}[\boldsymbol{a}]\, \boldsymbol{x}, \boldsymbol{y}\rangle. \tag{11}$$

The Lagrange multiplier combines the payoff function 0 with the constraints 11), taking into account the nonnegative multipliers λ, μ:

$$\mathcal{L}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}] \stackrel{\text{def}}{=} \mathcal{J}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}] + \lambda\left(\|\boldsymbol{x}\|_{2,m} - 1\right) + \mu\left(\|\boldsymbol{y}\|_{2,n} - 1\right). \tag{12}$$

The payoff function 0 with the conditions 0 has an equilibrium state $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, which can be calculated using the following equations:

$$\frac{\partial \mathcal{L}}{\partial x} \equiv \boldsymbol{A^T}[\boldsymbol{a}]\boldsymbol{y} + \lambda \boldsymbol{x} = \boldsymbol{0}, \quad \frac{\partial \mathcal{L}}{\partial y} \equiv \boldsymbol{A}[\boldsymbol{a}]\boldsymbol{x} + \mu \boldsymbol{y} = \boldsymbol{0}. \tag{13}$$

The solution to equation 0 is as follows:

$$\boldsymbol{A}[\boldsymbol{a}]\boldsymbol{A}^T[\boldsymbol{a}]\boldsymbol{y} = \lambda\mu\boldsymbol{y}, \quad \boldsymbol{A}^T[\boldsymbol{a}]\boldsymbol{A}[\boldsymbol{a}]\boldsymbol{x} = \lambda\mu\boldsymbol{x}. \tag{14}$$

Equations 0 on the left side contain two auxiliary matrices:

$$\mathcal{K}_A \stackrel{\text{def}}{=} \boldsymbol{A}^T[\boldsymbol{a}]\boldsymbol{A}[\boldsymbol{a}]\,, \quad \mathcal{K}_B \stackrel{\text{def}}{=} \boldsymbol{A}[\boldsymbol{a}]\boldsymbol{A}^T[\boldsymbol{a}]\,. \tag{15}$$

The matrix $\mathcal{K}_A$ is an m square Hermitian matrix. The matrix $\mathcal{K}_B$ is an n square Hermitian matrix. A real symmetric matrix is Hermitian. A Hermitian matrix is always self-adjoint. It follows from (Theorem 2.8, Section 2.4 in [14]), that both matrices $\mathcal{K}_A$ and $\mathcal{K}_B$ have the same nonzero eigenvalues, counting multiplicity. The matrices $\mathcal{K}_A$ and $\mathcal{K}_B$ are positive-semidefinite (Theorem 7.3, Section 7.1 in [14]). The number of zero eigenvalues of $\mathcal{K}_A$ and $\mathcal{K}_B$ is at least $|\mathrm{m}-\mathrm{n}|$. Let $\mathcal{K}_1$ be the matrix with the smallest dimensions of $\mathcal{K}_A$ and $\mathcal{K}_B$. In other words,

$$\mathcal{K}_1 \overset{\text{def}}{=} \begin{cases} \mathcal{K}_A & \text{if } m \le n, \\ \mathcal{K}_B & \text{if } m > n. \end{cases} \tag{16}$$

The matrix $\mathcal{K}_1$ is positive-semidefinite. The eigenvalues of matrix $\mathcal{K}_1$ are:

$$\begin{aligned} \mathbb{L}_i &= \text{eigenvalues}(\mathcal{K}_2) \ge 0, \\ i &= 1,..,\jmath, \\ \jmath &\overset{\text{def}}{=} \min(m,n). \end{aligned} \tag{17}$$

If $m = n$, then the matrices are equal $\mathcal{K}_A = \mathcal{K}_B$ and have the same set of eigenvalues. If $\mathcal{K}_1$ is positive definite, the number of its zero eigenvalues is exactly $|\mathrm{m}-\mathrm{n}|$. The eigenvalues of the positive definite matrix $\mathcal{K}_S$ are:

$$\begin{aligned} \mathbb{L}_i &= \text{eigenvalues}(\mathcal{K}_1) > 0, \\ i &= 1,..,\jmath, \\ \jmath &\overset{\text{def}}{=} \min(m,n). \end{aligned} \tag{18}$$

From Equation (12), it follows that $\lambda_i \mu_i = \mathbb{L}_i$. Finally,

$$\lambda_i = \mu_i = \sqrt{\mathbb{L}_i}, \qquad i = 1,..,\jmath. \tag{19}$$

Therefore, every matrix quadratic game has a balanced situation when it comes to mixed strategies. The proof of the theorem is constructive. That means it reduces the solution of the matrix game with $p = 2$ to a quadratic programming problem. Therefore, any way of solving linear programming problems can be used to solve matrix games.

The solution to the substratum matrix game with constrained actions (Equations 6, 8, and 10) is found. The goal of the Superstratum task is to find the best solution for the "designer" by looking for the extreme value of (14) by changing the design variables.

If players have the same resources, as shown in equation (8), the total payoff will be zero. In the event that player $\boldsymbol{x}$ possesses a greater quantity of resources than player $\boldsymbol{y}$, the possibilities available to them are subject to different constraints: $\|\boldsymbol{x}\|_{2,m} > \|\boldsymbol{y}\|_{2,n}$.

In this case, the "designer," who shares similar interests with the "operator," tries to reduce the maximum possible damage. The principal possible damage is shown by the major eigenvalue $\mathbb{L}_j$ with a negative sign. The goal is to make the largest eigenvalue, $\mathbb{L}_j[\boldsymbol{a}]$., as small as possible.

If player $\boldsymbol{x}$ has fewer resources than player $\boldsymbol{y}$, the following inequality holds: $\|\boldsymbol{x}\|_{2,m} < \|\boldsymbol{y}\|_{2,n}$. The designer will guarantee a specific outcome, regardless of the first player's actions, which are designated as "nature." [5]. The equilibrium point corresponds to the minimal positive eigenvalue $\mathbb{L}_1$. In this case, the goal of the "designer" is to make the minimal eigenvalue as large as possible, $\mathbb{L}_1[\boldsymbol{a}]$.

3. Antagonistic Stratified Games on Unit Square

3.1 Formulations of the stratified games on unit square

This section contains basic information about the theory of finite antagonistic games on a unit square. [10], [15] . This is a two-person game, also known as a zero-sum game, where the set of pure strategies for two opponent players is a interval $[0,\ 1]$. The two players are called "nature" and "operator." Any two-person zero-sum game that has a set of strategies for each player that is a continuum can be reduced to a game on a unit square. Games on the unit square are defined by pay-off functions $k_\alpha(x,\ y)$ that are defined on the unit square and functionally depend on the α as a parameter or a control function of the cardinal player ("designer").

3.2 Solutions of the substratum games on unit square

Players can use mixed strategies, which are functions that take arguments on the unit interval. The payoff of player "nature" when players "nature" and "operator" apply mixed strategies $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, is, by definition [16]:

$$\mathcal{J}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}] = \frac{1}{2} \int_0^1 \int_0^1 k_\alpha(t, \tau)\, dx(t)\, dy(\tau). \tag{20}$$

If $k_\alpha(x, y)$ is continuous in both arguments for all possible values of α, then

$$\max_x \min_y \mathcal{J}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}] = \min_y \max_x \mathcal{J}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}] = v_\alpha , \tag{21}$$

that is, for such a game 0 the minimax principle is valid and there exist a value for the game, denoted by v_α, and optimal strategies for both players. The value pf the game v_α depends functionally upon $\boldsymbol{\alpha}$. If $\boldsymbol{\alpha}$ is a scalar or vector parameter, the value of game is a function upon these parameters. Theorems about the existence of values in games (minimax theorems) have been proven under less strict assumptions about the payoff functions. The general minimax theorems imply that there is a value for games on a unit square with limited payoff functions. There are theorems that prove the value of a game in certain types of pay-off functions that are not continuous. However, not all games on the unit square have values.

The structure of the set for games with a unique solution has been determined for games on a unit square with continuous pay-off functions. It is important to note that the set of continuous functions in two variables for which the corresponding game on the unit square has a unique solution, for which the optimal strategies of both players are continuous, and whose supports are nowhere-dense perfect closed sets of Lebesgue measure zero contains an everywhere-dense subset.

It has been determined that there are no general methods for solving games on the unit square. However, for certain classes of games on the unit square, it is possible to find an analytical solution (e.g., for games of timing, games with pay-off functions depending only on the difference of the strategies of the two players, and having optimal equalizing strategies). Alternatively, it is possible to prove the existence of optimal strategies with finite support for such games, thereby reducing the problem of finding a solution for a game on the unit square to the solution of a matrix game. Approximate methods can be used to solve games with continuous payoff functions.

The strategy x of ordinal player 1 in the substratum game is an integrable function $\boldsymbol{x}$, which is constrained by the following equation:

$$\|\boldsymbol{X}\|_p = 1. \tag{22}$$

where the L^p norm is defined as:

$$\|\boldsymbol{X}\|_p \stackrel{\text{def}}{=} \left(\int_0^1 \mathrm{x}^p\, \mathrm{dt}\right)^{1/p}. \tag{23}$$

Similarly, the mixed strategy y of player 2 is an integrable function:

$$\|\boldsymbol{Y}\|_p = 1. \tag{24}$$

The positive natural number p in Eq. 0,0,0 determines the class of the game.

3.3 Solutions of the linear substratum games on unit square

The cited literature gives an overview of the methods used to evaluate game values. The usual setting $p = 1$ shows that the mixed strategy for ordinal players is just a probability distribution over their pure strategies. Any event must have a positive probability, and the total probability of all events must be one. Therefore, any strategy must meet the following conditions:

$$\begin{aligned} \|\boldsymbol{X}\|_1 &= 1, \\ \|\boldsymbol{Y}\|_1 &= 1. \end{aligned} \tag{25}$$

3.4 Solutions of the quadratic substratum games on unit square

If $p = 2$, there are Euclidean norms:

$$\|\boldsymbol{x}\|_2 \equiv \left(\int_0^1 \mathrm{x}^2\,\mathrm{dt}\right)^{1/2},$$
$$\|\boldsymbol{y}\|_2 \equiv \left(\int_0^1 \mathrm{y}^2\,\mathrm{d\tau}\right)^{1/2}. \quad (26)$$

This is a common situation in engineering and physical applications. In these situations, the actions available to the players are limited. The modules of strategy vectors are less or equal than two positive limit values:

$$\|\boldsymbol{x}\|_2 \leq \xi,$$
$$\|\boldsymbol{y}\|_2 \leq \eta. \quad (27)$$

The Lagrangian combines the pay-off function 0 with the constraints 0,0, taken with the non-negative multipliers λ, μ:

$$\mathcal{L}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}] \stackrel{\text{def}}{=} \mathcal{J}[\boldsymbol{x},\boldsymbol{y},\boldsymbol{a}] + \lambda(\|\boldsymbol{x}\|_2 - 1) + \mu(\|\boldsymbol{y}\|_2 - 1). \quad (28)$$

For the pay-off function 0 with the conditions 0, the equilibrium state $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ satisfies the equations:

$$\frac{\partial\mathcal{L}}{\partial x} \equiv \int_0^1 k_\alpha(t,\ \tau)\mathrm{d}\boldsymbol{y}(\boldsymbol{\tau}) + \lambda\boldsymbol{x} = \boldsymbol{0},$$
$$\frac{\partial\mathcal{L}}{\partial y} \equiv \int_0^1 k_\alpha(t,\ \tau)\mathrm{d}\boldsymbol{x}(\boldsymbol{t}) + \mu\boldsymbol{y} = \boldsymbol{0}. \quad (29)$$

The resolution of the Eq. 0 reads:

$$\int_0^1 \Phi(\tau,\rho)\mathrm{d}\boldsymbol{y}(\boldsymbol{\tau}) - \lambda\mu\boldsymbol{y} = \boldsymbol{0},$$
$$\int_0^1 \Psi(\tau,\rho)\,\mathrm{d}\boldsymbol{x}(\boldsymbol{\tau}) - \lambda\mu\boldsymbol{x} = \boldsymbol{0}. \quad (30)$$

If the kernel $k_\alpha(\tau,t) = k_\alpha(t,\tau)$ is symmetric, the transposed kernels match:

$$\Phi(\tau,\rho) = \int_0^1 k_\alpha(t,\tau)k_\alpha(t,\varrho)\boldsymbol{dt}\ , \quad (31)$$

$$\Psi(\tau,\rho) = \int_0^1 k_\alpha(\tau,t)k_\alpha(t,\ \varrho)\boldsymbol{dt}. \quad (32)$$

The Equations 0,0 have the same eigenvalues. The kernel $\Phi(\tau,\rho)$ is positive-semidefinite. The eigenvalues of the integral equation 0 with the kernel $\Phi(\tau,\rho)$ are:

$$\Lambda_i[\boldsymbol{a},\boldsymbol{\beta}] \stackrel{\text{def}}{=} \text{eigenvalue}(\Phi) \geq 0. \quad (33)$$

From Eqs. 0,0,0 follows, that $\lambda_i\mu_i = \Lambda_i$.
Finally, the values of Lagrange multipliers for each equilibrium point 0are equal to:

$$\lambda_i = \mu_i = \sqrt{\Lambda_i[\boldsymbol{a},\boldsymbol{\beta}]}. \quad (34)$$

As a result, every matrix game has equilibrium situations within the context of mixed strategies. Every equilibrium situation has a game value 0.

4. Game Formulation for a Potential Asteroid Impact

The following section will present the example of the stratified game in structural optimization. 65 million years ago, an asteroid triggered climate change. In 1908, an asteroid explosion in Siberia uprooted millions of trees. In Bavaria, an impact left behind a huge crater. Such disasters should be prevented in the future. The Earth is under constant bombardment from space: countless pieces of rock tumble through space and collide with everything in their path. Fortunately, major catastrophes are rare. But just how surprising the impact of a celestial body can be was demonstrated in 2013,

when the European Space Agency (ESA) joined forces with NASA on a double mission to investigate whether and how a dangerous asteroid could be deflected in an emergency. NASA's DART (Double Asteroid Redirection Test) probe was launched on November 24, 2021 to find out the effects of a deliberate impact with an asteroid. NASA wants to have the DART space probe hit it in the fall of 2022 and change its orbit. The experts call it a “cosmic collision”. The United Nations also wants to be prepared for an imminent collision with an asteroid. The organization is therefore setting up an international network to research defensive measures and create a functioning early warning system for the whole world. The international commission in charge of this is called the Space Mission Planning and Advisory Group. This Group develops a technical response strategy for a potential asteroid impact. Europe's space agency ESA is also trying to calculate the trajectory of potentially dangerous asteroids as early as possible. NEOCC coordination center for near-Earth objects collects information on discovered asteroids. Under the name NeoShield-2 [17], Europe also researched asteroid defense options for several years from 2012 under the leadership of the German Aerospace Center (DLR). The HERA mission is a direct result of this research, which recommended an experiment with such a “cosmic collision”. As outlined above, the motivation for the game formulation is provided by the cited international activities.
The probability of a "cosmic collision" is high, but the likelihood of it occurring is extremely low. However, such a case is not fully susceptible to prediction. The design task, when faced with conditions of aleatoric uncertainty, can be expressed as a stratified game. Namely, the “ordinal player” („nature”) operates on “substratum” and can apply an arbitrary admissible external action. The “cardinal player” („designer”) operates on the “superstratum” of the game and attempts to select the shape of the element $\boldsymbol{a}$. The control variables $\boldsymbol{a}$ and $\boldsymbol{p}$ are usually called strategies of the cardinal and ordinal player, respectively. The aim of the „designer” is to minimize the pay-off function $\mathcal{J}[\boldsymbol{a},\boldsymbol{p}]$, while “nature” tries to maximize it. Thus, the game is a zero-sum game because the sum of the payoffs to every player are the same for every single set of strategies. In these games, one player gains if and only if another player loses. Consequently, the interests of “designer” and “nature” are in conflict. This game is called antagonistic. It is important, that the design is characterized by a control set $(\boldsymbol{a},\boldsymbol{q})$, which represent the actions of both players.
The pay-off function $\mathcal{J}$ is the damage for the most unfavorable action of „nature”. The “nature” tries to upsurge the pay-off function, varying the components of vector $\boldsymbol{p}=\{p_1,p_2\}$. Contrarily, the „designer” weakens the pay-off function through the initial setting of the vector $\boldsymbol{a}=\{a_1,a_2\}$. The set of players is $\mathbf{I}=\{\boldsymbol{a},\boldsymbol{p}\}$. The pure strategy sets are $\mathbf{S}$ .Obviously, there is a conflict of the interests, which leads to the game formulation in normal form:

$$\mathbf{T}=\{\mathbf{I},\mathbf{S},\mathcal{J}\}. \tag{35}$$

It is evident that, within the framework of Euclidean geometry, the absolute value of a vector is equivalent to the square root of the sum of the squared components of that vector. The components of vector are restricted with 0,0,0 with $p=2$. The game formulation for a potential asteroid impact turn into the matrix quadratic game. In the continuous formulation, the integral damage in restricted quadratically with the integral constraint. The vector components will be restricted with the integral constraints 000 for $p=2$. The game's formulation, which initially addressed the potential impact of asteroids, was subsequently converted into a unit square game problem. It is evident that the "lower level" game could be extended. Specifically, the opposing 'ordinal' player, otherwise referred to as the 'pilot Pirks' [18], has the option to participate on the 'lower' level against the 'nature'. It is possible to formulate the corresponding game as the differential game.

5. Discussion

problem with the structure of a game is a problem that arises from external circumstances that are beyond the control of the parties involved in the game on a unit square. The scalar criterion works like the payoff function, or damage. The first players, called the "operator" and "nature," try to make the payoff as small or as big as possible on a unit square. They have to follow the rules of limited resources. The solution to the substratum game is expressed as a value of the game. The game's worth depends on how it's designed. The most important player, called the "designer," is in charge of the design rules. The goal of the game is to find the highest and lowest possible values. This is like a problem in design, where the best solution is to find the most optimal way to do things. If there are several important players with different goals, a superstratum game is created. This game is designed to cater to the superstratum players. The presentation will explain the best design problems for games that use a unit square. The suggested game plans can be used to design continuously controlled systems with uncertain loading in the best way.

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